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# Quantized stabilization of nonlinear affine systems

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# Quantized stabilization of nonlinear affine systems

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# DEDICATION

To my family

# ABSTRACT

Recently, the study of quantized control systems has attracted increasing attention by researchers, due to its theoretical and practical importance in hybrid control systems, control under communication/computation constraints, etc. This thesis is devoted to the problem of stabilizing nonlinear affine systems with quantized feedback. We show that, for a single-input nonlinear affine continuous-time system, a stabilizing quantizer can be constructed based on a control Lyapunov function, and a *robustly* stabilizing quantizer can be constructed based on a *robust* control Lyapunov function. We also characterize the coarsest quantizer under certain conditions. The quantized control scheme provides understanding to the problem of how much interaction between the controller and the system dynamics is needed for stabilization, and is furthermore useful for studying the interaction between control and information.

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# LIST OF ACRONYMS

| | |
|---|---|
| CLF | control Lyapunov function |
| ESLQ | essentially semi-logarithmic quantizer |
| FSLQ | finite semi-logarithmic quantizer |
| HSLQ | hierarchical semi-logarithmic quantizer |
| ISS | input-to-state stability |
| LHS | left hand side |
| LMI | linear matrix inequality |
| LQ | logarithmic quantizer |
| LTI | linear time-invariant |
| QCLF | quadratic control Lyapunov function |
| RCLF | robust control Lyapunov function |
| RCLP | robust control Lyapunov pair |
| RHS | right hand side |
| SLQ | semi-logarithmic quantizer |
| SVD | singular value decomposition |
| UCLF | uniform control Lyapunov function |
| w.r.t. | with respect to |

# 1 INTRODUCTION

This thesis focuses on studying the quantized stabilization of nonlinear affine systems. This work is an extension to previous research on quantized stabilization of linear systems in [16, 13]. We show that, for a single-input nonlinear affine continuous-time system, a stabilizing quantizer can be constructed based on a control Lyapunov function (CLF), and a *robustly* stabilizing quantizer can be constructed based on a *robust* control Lyapunov function (RCLF). We also characterize the coarsest quantizer under certain conditions. This research fits into the framework of investigating the complexity of the interaction between controllers and plant dynamics, and is useful for studying control systems with communication constraints and computational complexity.

This introduction is divided into three sections. Section 1.1 introduces briefly the notions relevant to quantized control. Section 1.2 describes the motivation for studying quantized control systems. Section 1.3 provides a summary of the chapters that follow.

## 1.1 Preliminaries of Quantized Control

In this section we present a brief introduction to quantized control. This helps the reader to intuitively understand quantized control before we give the precise definition.

A *quantizer* (or a quantized controller) is a controller that maps the states of a system into piecewise constant control inputs which take values in an at most countable set [8, 16]. In other words, in a quantized control system, the controller generates the control inputs based on the quantized information (incomplete knowledge) of the system states. The quantizer induces a partition of the state-space into an at most countable number of *cells*, each of which is associated with one control value (see Figure 1.1 for an example of a memoryless time-invariant quantizer). If, for example, the state of the system is in cell $\Omega_1$, then the associated value $u_1$ is employed as the control input. Each $u_i$ is called a *control primitive.*

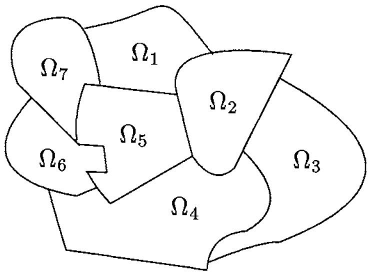


Figure 1.1 Illustration of a quantizer. The state-space is partitioned into cells, and cell $\Omega_i$ is assigned a fixed control value $u_i$.

The quantizer has certain nice features. It only needs to transmit/process information intermittently and with finite precision, in contrast to a traditional controller which must transmit/process information continuously and with infinite precision. As a consequence, the interaction between the controller and the plant exists only at discrete instants of time, and the interactions involve less information. Although less interaction is used during the control process, the quantizer may be designed to achieve stability or desired performance of the closed-loop system (see e.g. [16]). Therefore, quantized control is helpful to address some control problems, such as control under communication/computation constraints.

## 1.2 Motivation

Recently the problem concerning quantized control systems has attracted increasing attention by researchers; see [57, 44, 51, 8, 16] and references therein. The motivation for considering quantization in control systems comes from the observation that for many control systems, quantization is inevitable or useful.

In a continuous-variable control system, if a digital controller is used, or if the data transmission between the plant and the controller is constrained by digital communication channels, then quantization is inevitable. In this scenario, the effects of quantization are usually seen as undesirable, either as noise or state uncertainty, and must be reduced by often complex controllers.

However, another approach has a fundamentally different view, which could be theoretically and practically more attractive. Instead of viewing quantization as undesirable, in this scenario one intensionally introduces quantization and make use of it to address the problems of

1. Control under communication/computation constraints. As the control systems grow in size and in complexity and become distributed, a cost associated with communication/computation needs to be taken into account during the control process, whereas such a cost is neglected in traditional control theory. The solutions to this problem can be used to efficiently allocate communication/computation resources, or to effectively reduce attention cost [7], communication effort [24, 57, 10], and computational complexity [20].

2. The systematic way to design hybrid systems. Many hybrid phenomena (interaction between continuous dynamics and logic) are effects of information quantization. Therefore, quantization is considered to be useful for deriving systematic design methods for hybrid systems [13, 16].

3. The design of hierarchical systems. In hierarchical systems, it is evident that higher levels in the hierarchy manipulate only quantized information about the dynamics at lower levels [5]. Introducing quantization to a system may result in a hierarchical structure and is useful for designing a hierarchical system.

So far we have seen that quantization is useful in addressing several control problems. However, in this thesis we do not focus on these problems. Instead, we concentrate on a basic question concerning quantization: how can a system be stabilized by means of quantized control? It is obvious that the answer to this question has fundamental significance to the above problems. This thesis provides an answer to the question for nonlinear affine systems, whenever an RCLF or CLF is available.

## 1.3 Summary of the Thesis

This thesis is organized as follows.

**Chapter 2** In this chapter we review the relevant existing literature.

**Chapter 3** In this chapter we introduce the mathematical preliminaries of quantization and the Lyapunov-based design approach. The notions of stability and robustness suitable for a quantized continuous-time control system are also described.

**Chapter 4** In this chapter we define the main problems we want to solve, and briefly present the main results.

**Chapter 5** In this chapter we construct quantization for single-input nonlinear affine systems based on the availability of an RCLF. Under certain conditions the coarsest quantizers are given. Finite quantizers are also obtained. We further show that several important classes of nonlinear affine systems, such as linear systems, feedback linearizable systems, locally linearizable systems, etc., fall into this category, and their special features can be used to derive more specific quantization results.

**Chapter 6** In this chapter we show that, for any nonlinear affine system with a CLF available, the system can be stabilized by a hierarchical quantizer. Finite quantizers are also obtained.

**Chapter 7** In this chapter we discuss briefly some interesting but unsolved topics relevant to the results we obtained, such as chattering-free quantizers and attention cost.

**Chapter 8** In this chapter we present several simulation results using the quantization theory we derived in this thesis.

**Chapter 9** In this chapter we conclude the thesis and present some interesting questions that will be the subjects of our future research work.

In summary, we present a strategy and results for quantized stabilization of nonlinear affine systems. The original work is concentrated in Chapters 4 through 8 and in the sections concerning semi-logarithmic quantizers (SLQ) and RCLF's in Chapter 3. The results obtained in this thesis may be helpful to investigate several important control problems, such as the design of hybrid systems, the interaction between control and information, etc.

# 2 LITERATURE REVIEW

This thesis is concerned with quantized stabilization of nonlinear affine systems. Relevant work can be classified into the following categories: 1) quantized stabilization and performance; 2) quantized stabilization and performance under communication/computation constraints; and 3) the discrete dynamical aspects of quantized systems.

## 2.1 Quantized Control Systems: Stabilization and Performance Problems

The work in this category lays a foundation for the problems of achieving desired stability or performance under communication/computation constraints, which are described in Section 2.2.

### 2.1.1 Uniform Quantization

[9] investigates the problem of stabilizing an unstable multi-input discrete-time linear system by means of quantized feedback. A uniform quantization with countable infinite rectilinear cells is considered, since it is widely used in information theory and in analog-to-digital converters. It is shown that the closed-loop behavior resulting from quantization of measurement is quite different (and more complicated) than that resulting from approximation of measurement. In such a quantized control system, asymptotic stabilization cannot be achieved, since higher and higher resolution (smaller size of quantization cells) is needed when the state is closer and closer to the origin.

The quantization used in [9] has drawbacks since it lacks resolution where resolution is needed, and it has fine resolution where such resolution is not needed. [8] fixes this problem by assuming it is possible to change the resolution (the sensitivity) of a uniform quantizer with only a finite number of cells. The evolution of the sensitivity of the quantizer is described by a dynamic equation, resulting in quantized dynamic feedback (*not* memoryless). Both continuous-time and discrete-time multi-input linear systems are examined, and global asymptotic stabilization by quantized feedback is achieved.

The approach in [8] is extended in [33] to nonlinear systems under the condition of *input-to-state stability* (ISS). This is further extended in preprints [34, 35] to allow arbitrary shape of cells.

### 2.1.2 Logarithmic Quantization

A natural theoretical question regarding quantization is: What is the *coarsest* allowable quantization? Using the Lyapunov-based approach, [16] shows that the coarsest quadratically stabilizing quantizer for a single-input discrete-time linear system follows a *logarithmic* law. [16] also obtains the coarsest space-time quadratically stabilizing quantizer for a single-input continuous-time linear system under uniform sampling. Furthermore, it shows that the finite truncation of the logarithmic quantizer (LQ) with an infinite number of cells guarantees *practical stability* of the closed-loop system.

[13] investigates bounded energy gain performance of quantized single-input linear systems. An approach based on $\mathcal{H}_\infty$ CLF is introduced, which is an extension of the approach in [16]. The coarsest quantizer, again, is found to be logarithmic.

[15] provides a lower bound of the minimum density of quadratically stabilizing quantizers for a two-input discrete-time linear system. This paper also shows that the optimal quantizer is radially logarithmic.

Following the approach in [15], [14] conducts extensive investigation on quantized two-input discrete-time linear systems. It provides more elaborate and more informative lower bounds than those in [15]. Surprisingly, [14] shows that a two-input system has quantization density no less than a single-input system, which contradicts our intuition that two inputs can do better than one.

The results listed so far in this chapter may cause *chattering* for continuous-time systems. Mathematically, chattering is allowed by *Filippov solutions* [18], but it is harmful practically. To prevent chattering, [24] introduces a *dwell-time* constraint to an LQ. It shows that there is a way to design an LQ with a dwell-time that asymptotically stabilizes a continuous-time linear system.

### 2.1.3 Other Work

[17] provides a rather complete analysis of the quantized stabilization of a discrete-time scalar (one-variable) linear system. An approach based on a possibly non-quadratic CLF is proposed, and a new lower bound on the number of cells is obtained. A logarithmic connection between the number of cells and the convergence time is established. [2] reports the results on the reachable sets of quantized linear systems, as well as on a class of nonlinear systems, i.e., nonholonomic chained-form systems. See also [32, 23] for other work.

## 2.2 Quantized Control Systems: Problems of Communication/ Computation Constraints

In this category, the communication/computation effort, or more generally, attention cost, needed to stabilize a control system is considered.

[57, 58] initiate a systematic research on control under communication constraints. The measurements are assumed to be coded and transmitted over a digital communication channel with finite capacity, which introduces quantization. The state estimation problem is explored in [57]. A weaker stability (called *containability*) of the closed-loop system using a coded feedback control law is studied in [58].

[7] investigates the *attention cost* for a control law. A control law with a lower cost needs less control effort; i.e., it is easier to implement. Quantized control (piecewise constant control) is regarded as easy to implement. An attention functional is proposed,

which, together with the performance functional, serves as the objective functional we need to minimize. The minimizing controller achieves a balance between the performance of the system and the difficulty involved in implementing the control.

[36] investigates nonlinear feedback systems and the control effort of quantized feedback in a stochastic scenario. The attention cost is defined to be the mean number of switchings per unit time for a quantized control law. This paper provides an explicit formula of the attention cost and conditions under which a system can have a finite attention cost.

[41, 42, 43] study the problem of state estimation via a finite capacity channel. [41] provides a coding-estimation scheme that generates an asymptotic mean-square error of zero under some conditions. [42, 43] provide the optimal and suboptimal coder-estimator structures. [44] considers communication-limited stabilization for a discrete-time linear system. It is shown that the necessary data-rate for stabilization depends only on the unstable poles.

[10] also examines the state estimation problem. A recursive estimation algorithm is proposed. This algorithm is computationally non-expensive and easy to implement in real-time systems.

[51, 52] perform an extensive research on the problem of control under communication constraints. The *minimum data-rates* needed for state estimation and feedback stabilization are studied, and they are shown to depend only on the unstable poles of the discrete-time linear system. A coding scheme for a channel with feedback is provided.

[45, 46] reexamine the notion of channel capacity in situations where the communication channel is a part of a feedback loop. It is shown that Shannon capacity is insufficient. A concept called *any-time capacity* is introduced.

[20, 19] intensionally introduce quantization to control systems to reduce the computation effort. It is shown that the quantization of a highly nonlinear system with symmetry leads to a relatively simple control architecture based on a hybrid automaton, which is suitable for real-time application.

See also [53, 54, 22] for other related work.

## 2.3 Quantized Control Systems: Problems of Discrete Dynamical Aspects

In this category, the hybrid features, especially the discrete (or symbolic, logical, qualitative) dynamical aspects of quantized control systems are stressed.

[5, 6] investigate the interaction between discrete dynamics and continuous dynamics. Quantization results from this interaction. Relevant work includes [11, 12].

[38] considers the discrete-event representation of a quantized linear system. In such a system the output of the quantizer behaves like a discrete-event system. Sufficient conditions are given to ensure that the discrete-event behavior is deterministic. [37] deals with the diagnosis of a quantized system through its discrete-event representation.

## 2.4 Summary

In this chapter we have reviewed literature relevant to quantized control systems. The first category of the literature focuses on the problem of achieving stability or desired performance for a quantized control system without considering communication/computation constraints explicitly. Our research work on quantized stabilization of nonlinear affine systems falls into this category, and is an extension to the existing work [16]. The second category includes research on achieving stability or desired performance for a quantized control system with communication/computation constraints. The third category investigates the discrete dynamical aspects of quantized systems.

# 3 MATHEMATICAL PRELIMINARIES

In this chapter we present the mathematical preliminaries for quantized control systems. These preliminaries include: 1) the precise definitions of various quantizers and their properties; 2) the Lyapunov-based approach and the RCLF; and 3) the notions of stability and robustness suitable for a quantized continuous-time control system.

In this thesis we mainly consider the single-input nonlinear affine continuous-time system

$$\dot{x} = f(x) + g(x)u; \quad f(0) = 0 \tag{3.1}$$

where $f$ and $g$ are $\mathcal{C}^1$ functions, $x \in X \subseteq \mathbb{R}^n$, $X$ is the state-space, $u \in \mathbf{U} \subseteq \mathbb{R}$, and $\mathbf{U}$ is the admissible control set.

## 3.1 Introduction to Quantized Control

In this section we give the precise definitions of the quantizer, logarithmic quantizer, semi-logarithmic quantizer, and finite quantizer. Although special structures are imposed on the definitions of the quantizers (such as symmetry w.r.t. the origin, etc.), they do not lead to restriction; as we will see later that these structures arise naturally in the solutions of the problems considered in this thesis. The properties of quantizers are also discussed in this section.

### 3.1.1 Definition of Quantizer

A quantizer is a controller that maps the states of a system into piecewise constant control inputs which take values in an at most countable set.

**Definition 3.1.** *A (memoryless, time-invariant) quantizer is a 4-tuple* $(q, S, \Omega, \mathcal{U})$ *consisting of a map* $q : S \to \mathcal{U}$ *such that* $q(0) = 0$ *and for any* $i \in \mathbb{Z}$, $q(x) = u_i$ *if* $x \in \Omega_i$*; a set* $S \subseteq X$ *containing a neighborhood of the origin; a disjoint partition* $\Omega = \{\Omega_i\}_{i=-\infty}^{\infty}$ *of* $S$*; and a set of admissible control* $\mathcal{U} = \{u_i \in \mathbb{R}, i \in \mathbb{Z}\}$. *Every* $\Omega_i$ *is called a cell;* $i$ *is called the index of the cell. With a slight abuse of terminology,* $q$ *is called the quantizer.*

When we say quantizer in this thesis, we always refer to a memoryless time-invariant quantizer unless otherwise specified. The quantizer induces a partition of the state-space into an at most countable number of cells. Each cell $\Omega_i$ is associated with a fixed control input $u_i$. $\mathcal{U}$ forms a set of *control primitives.* If, for example, the state of the system is in some cell $\Omega_i$, then $u_i$ is employed as the control input. See Figure 1.1 for an example of a quantizer in a 2-D state-space.

### 3.1.2 Definitions of (Semi-)Logarithmic Quantizers

Logarithmic quantization, as achieved or used in [16, 13, 24], captures the intuition that the farther from the origin the state is, the less precise the control action and knowledge about the location of the state need to be. It is shown in [16] that the log quantization is the solution to a certain linear quantization problem.

**Definition 3.2.** *A* $\rho$*-based logarithmic quantizer (LQ) is a quantizer* $(q, S, \Omega, \mathcal{U})$ *with* $q$ *such that for any* $i \in \mathbb{Z}$, $q(x) = u_i$ *if* $x \in \Omega_i^+$, $q(x) = -u_i$ *if* $x \in \Omega_i^-$, *and* $q(x) = 0$ *if* $x \in \Omega_{zero}$, *with* $\Omega$ *being given as*

$$\begin{aligned} \Omega_i^+ &= \{x \in S | \gamma_{i+1} < \wp' x \leq \gamma_i\} \quad \forall i \in \mathbb{Z} \\ \Omega_i^- &= \{x \in S | -\gamma_i \leq \wp' x < -\gamma_{i+1}\} \quad \forall i \in \mathbb{Z} \\ \Omega_{zero} &= \{x \in S | \wp' x = 0\} \end{aligned} \tag{3.2}$$

*and with* $\mathcal{U} = \{\pm u_i | u_{i+1} = \rho u_i, i \in \mathbb{Z}\} \cup \{0\}$, *where* $0 \leq \rho < 1$ *is called the base,* $\wp \in \mathbb{R}^n$ *is a constant vector, and* $\gamma_{i+1} = \rho \gamma_i$, $i \in \mathbb{Z}$. *For cell* $\Omega_i^+$ *(or* $\Omega_i^-$, $\Omega_{zero}$*), the index is* $i^+$ *(or* $i^-$, *zero, respectively).*

Note that $\wp' x$ gives the scalar product of vectors $\wp$ and $x$, and the boundaries of cells are in fact the level surfaces of the linear function $\wp' x$.

For nonlinear systems, it is often convenient to consider semi-log quantizers, which have the major properties of LQ's. In fact, we will see that the semi-log quantization is the solution to a certain nonlinear quantization problem.

**Definition 3.3.** *A $\rho$-based semi-logarithmic quantizer (SLQ) is a quantizer $(q, S, \Omega, \mathcal{U})$ with $q$ such that for any $i \in \mathbb{Z}$, $q(x) = u_i$ if $x \in \Omega_i^+$, $q(x) = -u_i$ if $x \in \Omega_i^-$, and $q(x) = 0$ if $x \in \Omega_{zero}$, with $\Omega$ being given as*

$$\begin{aligned} \Omega_i^+ &= \{x \in S | \gamma_{i+1} < p(x) \leq \gamma_i\} \quad \forall i \in \mathbb{Z} \\ \Omega_i^- &= \{x \in S | -\gamma_i \leq p(x) < -\gamma_{i+1}\} \quad \forall i \in \mathbb{Z} \\ \Omega_{zero} &= \{x \in S | p(x) = 0\} \end{aligned} \tag{3.3}$$

*and with $\mathcal{U} = \{\pm u_i | u_{i+1} = \rho u_i, i \in \mathbb{Z}\} \cup \{0\}$, where $0 \leq \rho < 1$ is called the base, $p : S \longrightarrow \mathbb{R}$ is a smooth function with $p(0) = 0$, $p$ is called the partition function, and $\gamma_{i+1} = \rho\gamma_i$, $i \in \mathbb{Z}$.*

*If a quantizer defined on $S$ is $\rho$-based semi-logarithmic except on a finite number of cells, it is called an essentially semi-logarithmic quantizer (ESLQ) with base $\rho$.*

**Remark 3.1.** It is shown in [16, Lemma 2.1] that for an LQ, the choice of $\gamma_0$ is immaterial in considering stabilization problems. In Lemma A.1 (see Appendix A) we can show this is also true for an SLQ. Therefore, we assume $\gamma_0$ to be 1 without loss of generality in this thesis.

Note that the only difference between an LQ and an SLQ is that the linear function $\wp' x$ for an LQ is replaced by a smooth function $p : S \longrightarrow \mathbb{R}$ with $p(0) = 0$. Therefore, log partition has rectilinear boundaries (which are the level surfaces of the linear function $\wp' x$), and semi-log has curve boundaries (which are the level surfaces of the function $p(x)$). Log partition is symmetric w.r.t. the origin in the state-space, and each cell is a connected set, whereas semi-log is in general not symmetric, and each cell is not necessarily a connected set. See Fig. 3.1 for log and semi-log partitions in a 2-D state-space.

For an SLQ, since both the control value $u$ and the partition of $p(x)$ follow a logarithmic law with the same base, the graph of the quantizer in the $p(x)$-$u$ plane is self-similar

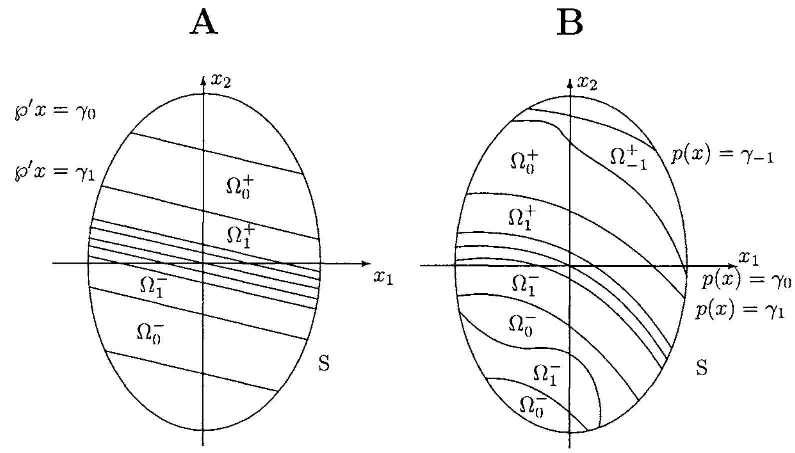


Figure 3.1 Examples of log and semi-log partitions in a 2-D state-space. **A** shows an LQ defined on $S$, and **B** shows an SLQ defined on $S$.

with similarity ratio $\rho$. (To be more precise, the graph is constructed by homothety with ratio $\rho$ [55]. See Figure 3.2.) Note that although the partition is generally not symmetric w.r.t. the origin in the state-space for an SLQ, the quantizer *is* symmetric w.r.t. the origin in the $p(x) - u$ plane, as is an LQ.

The self-similarity of an SLQ implies that its structure can be constructed using only four elements $S$, $\rho$, $p(x)$, and $u_0$. Once the four elements are specified, the SLQ is uniquely defined as follows:

1. Use $S$, $p(x)$, and $\rho$ to generate the partition $\Omega$ by equation (3.3).

2. Use $u_0$ and $\rho$ to generate the set $\mathcal{U}$.

3. $q$ is defined to map from the state $x$ in each cell to its associated $u$.

Each of these steps are easy to perform. Therefore, in practice it is easy to store the data about the four elements in memory and do calculation online to generate the quantizer.

Notice that the cells become larger and larger when $p(x)$ is farther away from 0, as for LQ's described in [16]. Notice again that, whenever the state $x$ is approaching the boundary of a cell, the corresponding point in the $p(x)$-$u$ plane is either approaching

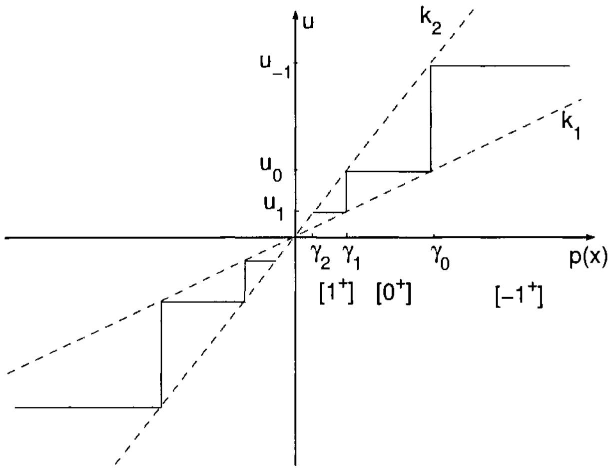


Figure 3.2 The graph of semi-log partition in the $p(x)$-$u$ plane. The graph has a self-similar and symmetric structure. Each number in brackets is the index of the cell. For any $x$ s.t. $p(x) \in (\gamma_1, \gamma_0]$, index $0^+$ is transmitted and $u_0$ is used as the control input.

the line $u = k_1 p(x)$, or approaching the line $u = k_2 p(x)$. In the $x$-$u$ space, $u = k_1 p(x)$ and $u = k_2 p(x)$ are two manifolds, which are called the *triggering manifolds.* It is easily seen that an SLQ can be uniquely determined by its two triggering manifolds. For LQ's, the two triggering manifolds are simply two subspaces in the $x$-$u$ space, which makes its implementation easy.

A system with an SLQ can be seen as an automaton. The automaton has a countable infinite number of states, with a fixed output (i.e., the control input of the plant) assigned to each of them. Each state of the automaton is associated with one cell in the system's state-space. An instantaneous transition to a different state takes place if $x$ crosses the boundary of a cell, and the new state of the automaton is decided by the position of $x$, i.e., the index of the cell that $x$ is entering. As the system evolves continuously, the automaton evolves at discrete instants of time, and generates corresponding control inputs. Figure 3.3 illustrates the state transition of an automaton.

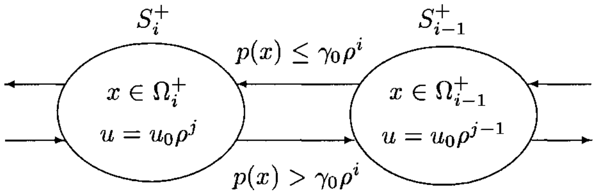


Figure 3.3 A hybrid automaton for a system with an SLQ. $S_i^+$ denotes the state of the automaton associated with cell $\Omega_i^+$, etc.

### 3.1.3 Definition of Finite Quantizer

Below, we introduce finite quantizers. In practice, finite quantizers are used instead of infinite quantizers, since finite quantizers require only finite precision whereas infinite quantizers require infinite precision when the state approaches the origin.

**Definition 3.4.** *A finite quantizer (of order N) is a quantizer q with* $\Omega = \{\Omega_i\}_{i=-N+1}^{N-1}$, *and* $\mathcal{U} = \{u_i \in \mathbb{R} | u_i = -u_{-i}, i = -N+1, \cdots, 0, \cdots, N-1\}$.

The finite quantizer used in this thesis is normally a finite truncation of an SLQ, which is obtained as follows. Consider an SLQ as defined by Definition 3.3. For some $j \in \mathbb{Z}$, let $\Omega_*^+ = \{x \in S | 0 < p(x) \le \gamma_j\}$, and $\Omega_*^- = \{x \in S | 0 > p(x) \ge -\gamma_j\}$, and use $u^* = k_1\gamma_j$ in $\Omega_*^+$, $-u^*$ in $\Omega_*^-$. For any $i < j$, let $\Omega_i^\pm$ and the corresponding $u$ be as in Definition 3.3. Then this quantizer has a finite number of cells. We refer to this type of finite quantizer as a *finite semi-logarithmic quantizer* (FSLQ).

### 3.1.4 Density of Quantizer

We define the density of a quantizer as follows.

**Definition 3.5.** *For a quantizer q of system (3.1), let* $0 < \epsilon \le 1$, *and let* $\#q^+[\epsilon]$ *and* $\#q^-[\epsilon]$ *denote the numbers of control values that* $\mathcal{U}$ *has in the intervals* $[\epsilon, \frac{1}{\epsilon}]$ *and* $[-\frac{1}{\epsilon}, -\epsilon]$, *respectively. Define*

$$\eta_q = \max\{\limsup_{\epsilon \to 0} \frac{\#q^+[\epsilon]}{-\ln \epsilon}, \limsup_{\epsilon \to 0} \frac{\#q^-[\epsilon]}{-\ln \epsilon}\}.$$

*$\eta_q$ is called the quantization density of $q$. For two quantizers $f$ and $g$, $f$ is said to be coarser than $g$ if $\eta_f < \eta_g$.*

A $\rho$-based SLQ $q$ defined on $\mathbb{R}^n$ has density $\eta_q = \frac{2}{\ln \frac{1}{\rho}}$. A $\rho$-based SLQ $q$ defined on a compact set in $\mathbb{R}^n$ has density $\eta_q = \frac{1}{\ln \frac{1}{\rho}}$. A finite quantizer has density zero. Two quantizers which are only different in a finite number of cells have the same density, so a $\rho$-based ESLQ has the same density as a $\rho$-based SLQ.

## 3.2 Introduction to the Lyapunov-based Design

In this thesis we will construct quantizers for system (3.1) based on the availability of a CLF for (3.1). Below we introduce the the Lyapunov-based design. In this thesis smoothness is assumed for all CLF's.

### 3.2.1 Definition of CLF

We introduce the concept of a CLF for a single-input nonlinear control system

$$\dot{x} = F(x, u); \quad F(0, 0) = 0 \tag{3.4}$$

where $x \in X$, $u \in \mathbf{U}$ are defined as before. $F$ is continuous.

A function $V : \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$ is said to be *positive definite* if $V(X) > 0$ for all $x \neq 0$, and $V(0) = 0$. It is said to be *proper* if $\{x | V(x) \leq a\}$ is compact for all $a > 0$. A smooth function $V$ is said to be *infinitesimally decreasing* if for any compact set $E \subseteq X$, there is some compact subset $U \subseteq \mathbf{U}$ and for each $0 \neq x \in E$, we have

$$\inf_{u \in U} \langle DV, F(x, u) \rangle < 0. \tag{3.5}$$

Here $\langle \cdot, \cdot \rangle$ denotes the inner product, and $DV = (\frac{\partial V}{\partial x})'$. It can be shown that (3.5) is equivalent to the existence of a positive definite continuous function $W : \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$, such that

$$\inf_{u \in U} \langle DV, F(x, u) \rangle < -W(x). \tag{3.6}$$

**Definition 3.6.** *A control Lyapunov function (CLF) for system (3.4) is a function with the properties of positive definiteness, properness, and infinitesimal decrease.*

For affine system (3.1), (3.5) is equivalent to

$$L_gV = 0 \Longrightarrow L_fV < 0 \tag{3.7}$$

for each $0 \neq x \in X$. Here $L_gV(x) = \frac{\partial V}{\partial x}g$, and $L_fV(x) = \frac{\partial V}{\partial x}f$.

We call a feedback $k$ *regular* if it is locally Lipschitz on $X \setminus \{0\}$. The properties of CLF's are listed as follows [49, 48]:

**Lemma 3.1.** *(a) For system (3.1), it admits a regular stabilizing feedback if and only if it admits a CLF. (Artstein's Theorem)*

*(b) For system (3.4), the existence of a CLF implies asymptotic controllability of the system.*

*(c) For system (3.4), any regular feedback such that a CLF is strictly decreasing is asymptotically stabilizing.*

### 3.2.2 Definition of Uniform CLF

Now we introduce the concept of a uniform CLF (UCLF) for a single-input control system under persistently acting disturbances

$$\dot{x} = F_d(x, u, d); \quad F_d(0, 0, 0) = 0 \tag{3.8}$$

where $x \in X, u \in \mathbf{U}$ are defined as before, $F_d$ is continuous, the disturbance $d(\cdot)$ is a scalar measurable function taking values in $D$, $D$ is a compact set of admissible disturbance, $d_M = \max_{d \in D} |d|$.

**Definition 3.7.** *Let $V : \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$ be smooth, positive definite, and proper. $V$ is said to be a uniform CLF (UCLF) for system (3.8) if there exists a continuous positive definite function $W : X \rightarrow \mathbb{R}_{\geq 0}$, and for any bounded set $E \subseteq X$, there is some compact subset $U \subseteq \mathbf{U}$ such that*

$$\min_{u \in U} \max_{d \in D} \langle DV, F_d(x, u, d) \rangle < -W(x) \quad \forall x \in E, x \neq 0. \tag{3.9}$$

Roughly speaking, a UCLF is a CLF whose derivative can be made negative pointwise by the choice of control value for any admissible disturbance $d$. A fundamental result regarding UCLF's is that the control system (3.8) admits a UCLF if and only if there exists a robustly stabilizing feedback for it [30, Theorem 1]. The precise statement of this result and the notion of *robust stabilization* will be given in Section 3.3. For details about UCLF's, we refer to [30, 31].

### 3.2.3 Definition of RCLF

In this subsection, we first present the definition of an RCLF and then show it does guarantee certain robustness. The properties of an RCLF are also described. The concept of RCLF is crucial for deriving the results in this thesis. In later chapters we will see that the availability of an RCLF guarantees the existence of a finite density robustly stabilizing quantizer, and the quantizer design method developed for RCLF's extends to a method for CLF's when only a CLF is available.

**Definition 3.8.** *$(V(x), \alpha)$ is called a robust control Lyapunov pair (RCLP) for system (3.4) on a compact set $S \subseteq X$ containing a neighborhood of 0 if $\alpha > 0$, $V(x)$ is a CLF for (3.4) on $S$, and there exists some admissible control $u_x$ for each $x \neq 0$ in $S$, such that*

$$\alpha^2 u_x^2 + \langle DV, F(x,u) \rangle < 0. \tag{3.10}$$

The $V(x)$ in the above definition is called a *robust control Lyapunov function* (RCLF) for (3.4) on $S$. For simplicity, we always assume without loss of generality that $S$ is a closed invariant set of $V(x)$ unless otherwise specified.

The RCLF is so called since it guarantees the existence of robustly stabilizing feedback (to be elaborated in Section 3.3). In the next lemma we show that an RCLF for the undisturbed system (3.4) is a UCLF for the system with a small enough persistently acting disturbance

$$\dot{x} = F(x,u) + G(x,u)d \tag{3.11}$$

where $F(0,0) = G(0,0) = 0$, $F$ and $G$ are continuous. Furthermore, we assume $\|G(x,u)\|/u^2$ is bounded by a constant $c$ on the set $S \times \mathbf{U}$ (where $\|\cdot\|$ is the norm induced by $\langle\cdot,\cdot\rangle$). This assumption implies that the effect of the disturbance $d$ cannot dominate the control input $u$; otherwise the system may not be controllable from $u$.

**Lemma 3.2.** *Suppose $(V,\alpha)$ is an RCLP for the undisturbed system (3.4) on $S$, and $V_M = \max_{x\in S}\|DV\|$ (where $\|\cdot\|$ is the norm induced by $\langle\cdot,\cdot\rangle$). $V$ is a UCLF for (3.11) on $S$ if $\alpha^2 > cV_M d_M$.*

*Proof.* We only need to show that if for any compact set $E \subseteq S$, and for each $x \in E$, $x \neq 0$, we have

$$\max_{d\in D}\langle DV, F(x,u_x) + G(x,u_x)d\rangle < 0.$$

Because $\|G(x,u)\|/u^2$ is bounded by $c$, by the Cauchy-Schwarz Inequality, we know

$$\langle DV, G(x,u_x)d\rangle \leq cV_M|d|u_x^2.$$

Thus, for any $x \in E$, $x \neq 0$,

$$\begin{aligned}
&\max\nolimits_{d\in D}\langle DV, F(x,u_x) + G(x,u_x)d\rangle \\
&= \langle DV, F(x,u_x)\rangle + \max\nolimits_{d\in D}\langle DV, |G(x,u_x)|d\rangle \\
&< -\alpha^2u_x^2 + cV_M d_M u_x^2 \\
&< 0.
\end{aligned}$$

Therefore, $V$ is a UCLF for (3.11). □

This lemma says in essence that if the derivative of a CLF for an undisturbed system can be made negative enough pointwise by the choice of control input, then it is a UCLF for a disturbed system if the disturbances are small enough, and therefore robustly stabilizing feedback exists. Notice that a larger $\frac{\alpha^2}{V_M}$ implies more robustness of the closed-loop system for a given $V$. If we normalize $V_M$ to be 1, then $\alpha$ can measure the robustness of the closed-loop system. That is, if $(V_1,\alpha_1)$ and $(V_2,\alpha_2)$ are two RCLP's with $V_{M1} = V_{M2} = 1$, and if $\alpha_1 > \alpha_2$, then $(V_1,\alpha_1)$ guarantees more robustness than $(V_2,\alpha_2)$. Hence, we call $\alpha$ the *robustness level* if $V_M$ is normalized. Note this definition of an RCLF differs from that in [21].

Concepts similar to our definition of RCLF are inspected by several researchers. In [47, 26] a generalized form of equation (3.10) is used to achieve optimality and robustness to a class of uncertainties. In [4] it is used to render finite gain (from $x$ to $u$) at the origin. An RCLF ensures finite gain at the orgin since equation (3.10) gives penalty to using large control when it requires $V$ to decrease.

Several classes of nonlinear affine systems, such as linear systems, feedback linearizable systems, and locally linearizable systems, admit RCLF's. They are to be studied in greater detail in Chapter 5.

Suppose $V$ is a CLF for system (3.1) on $S$. Define $h(x) = \frac{(L_gV)^2}{4L_fV}$ on the set $S_f = \{x \in S | L_fV(x) > 0\}$. Let $\alpha_M^2 = \inf h(x)$, and $\alpha_S^2 = \liminf_{x \to 0} h(x)$. Then an RCLF has the following properties.

**Lemma 3.3.** *Suppose $V$ is a CLF for system (3.1) on $S$. The following statements are equivalent:*

*(1) $V$ is an RCLF for system (3.1) on $S$.*

*(2) $V$ is such that $\alpha_M^2 > 0$.*

*(3) $V$ is such that $\alpha_S^2 > 0$.*

*(4) $V$ is such that $u = kL_gV(x)$ is stabilizing for some constant $k < 0$.*

*Proof.* (1) $\Rightarrow$ (2): Suppose $V$ is an RCLF on $S$. Then there is some $\alpha^2 > 0$ such that for each $x \neq 0$ in $S$, we can find some $u_x \in \mathbf{U}$ so that (3.10) holds. Therefore,

$$\Delta = (L_gV)^2 - 4\alpha^2 L_fV > 0 \quad \forall x \in S \setminus \{0\}. \tag{3.12}$$

So $0 < \alpha^2 < h(x)$ on $S_f$. Since $\alpha_M^2$ is the infimum of $h(x)$, we know $\alpha_M^2 \geq \alpha^2 > 0$.

(2) $\Rightarrow$ (1): Suppose $\alpha_M^2 > 0$. For any $0 < \alpha^2 < \alpha_M^2$, if $L_fV < 0$, then $\Delta > 0$; if $L_fV = 0$ and $x \neq 0$, then $L_gV \neq 0$ and hence $\Delta > 0$; if $L_fV > 0$, then the choice of $\alpha$ makes $\Delta > 0$. Therefore, (3.12) is true, so (3.10) holds. Thus $V$ is an RCLF.

(2) $\Rightarrow$ (3): It is obvious that $0 < \alpha_M^2 \leq \alpha_S^2$ since $\alpha_M^2$ is the infimum of $h(x)$ on $S_f$.

(3) $\Rightarrow$ (2): Notice that on the set $\mathrm{cl}(S_f)$ ($\mathrm{cl}(\cdot)$ denotes the closure), $h(x)$ can be zero only if $x = 0$. So $\alpha_S^2 > 0$ implies $\alpha_M^2 > 0$; otherwise we would have $\alpha_S^2 = 0$.

(2) $\Rightarrow$ (4): Take $k = -\frac{1}{4\alpha^2}$ for any $0 < \alpha^2 < \alpha_M^2$. Let $u(x) = kL_gV(x)$. Then for any

$x \in S$,

$$\begin{aligned} &\alpha^2 u^2 + L_g V u + L_f V \\ &= \alpha^2 \tfrac{(L_g V)^2}{4\alpha^4} - \tfrac{(L_g V)^2}{2\alpha^2} + L_f V \\ &= -\tfrac{(L_g V)^2}{4\alpha^2} + L_f V \\ &< 0. \end{aligned} \tag{3.13}$$

The last inequality is true since, if $L_f V > 0$ then $\alpha^2 < \frac{(L_g V)^2}{4 L_f V}$, and if $L_f V = 0$ then $L_g V \neq 0$, and if $L_f V < 0$ then it is automatically satisfied. So $u = k L_g V(x)$ is stabilizing.

(4) $\Rightarrow$ (2): Suppose for any $x \neq 0$, $L_f V + k(L_g V)^2 < 0$ for some constant $k < 0$. Then if $L_f V > 0$, we have $\frac{(L_g V)^2}{L_f V} > -\frac{1}{k}$. So $\alpha_M^2 > 0$. □

Condition (2) says that the problem of checking when a CLF is an RCLF is reduced to solving a constrained optimization problem. Condition (3) says that $V$ is an RCLF if and only if it has a certain limiting property around the origin. Condition (4) says that $V(x)$ is an RCLF if and only if *domination redesign* (cf. [26]) is applicable. These conditions may be difficult to check. For checking when a CLF is an RCLF, the following lemma provides a sufficient condition [26]:

**Lemma 3.4.** *Suppose $V(x)$ is a CLF for system (3.1) on $S$. Suppose further the first nontrivial terms in the Taylor expansions of $f(x)$, $g(x)$, and $V(x)$ have degrees $d_f$, $d_g$, and $d_V$, respectively, and let $f_h(x)$, $g_h(x)$, and $V_h(x)$ denote these nontrivial terms. If $V_h$ is a CLF for the system $\dot{x} = f_h(x) + g_h(x)u$ and if $d_V \leq d_f - 2d_g$, then $V(x)$ is an RCLF for system (3.1) on $S$.*

Once we have verified that a CLF is indeed an RCLF, then $(V(x), \alpha)$ is an RCLP if $0 < \alpha^2 < \alpha_M^2$.

We define the coarsest quantizer, or the quantizer with the smallest density, as follows.

**Definition 3.9.** *Given an RCLP $(V(x), \alpha)$ for system (3.1) on $S$, $\alpha^2 < \alpha_M^2$, let $\mathcal{Q}_\alpha(V)$ denote the set of all quantizers $q(x)$ such that for any $x \in S$, $x \neq 0$,*

$$\alpha^2 (q(x))^2 + L_g V(x) q(x) + L_f V(x) < 0. \tag{3.14}$$

*A quantizer $q$ is said to be the coarsest for $(V(x), \alpha)$ if $q \in \arg\inf_{g \in \mathcal{Q}_\alpha(V)} \eta_g$.*

The set $\mathcal{Q}_\alpha(V)$ is in fact the set of quantizers that robustly stabilizes system (3.1). A quantizer which is the coarsest for $(V(x), \alpha)$ needs not be unique since different sets $\mathcal{U}$ may satisfy the same asymptotic property, and for the same $\mathcal{U}$ there may be different ways to define the function $q$ mapping $S$ into $\mathcal{U}$. Moreover, a quantizer which is the coarsest for $(V(x), \alpha)$ may not be an element of $\mathcal{Q}_\alpha(V)$. At any rate, the density of quantization induces a measure of coarseness on the partitions in the state-space. The smallest density of all robustly stabilizing quantizers can be seen as an indicator of the complexity of the interaction between the controller and the plant. (Note the smallest density here depends on how "good" the RCLF is.)

## 3.3 Discontinuous Systems, Stability, and Robustness

Quantized feedback is a discontinuous feedback. Once a discontinuous feedback control $k(x)$ is employed in a continuous-time control system, the existence and uniqueness of solutions, the notions of stability and robustness, and related theorems need to be reexamined or restated. In this section we provide some preliminaries for a discontinuous system.

In this thesis the solutions to a quantized control system as well as the stability and robustness are to be interpreted according to [50, 31]. It can be shown that quantizers obtained in this thesis guarantee the existence of solutions, but not uniqueness. Based on the properties of CLF's shown in [50, 30, 31], we can prove the following useful lemma connecting (robust) CLF's to (robust) stabilization of discontinuous systems:

**Lemma 3.5.** *(a) Suppose $V$ is a CLF for system (3.4). Then if $k(x)$ is such that*

$$\langle DV, F(x, k(x)) \rangle < 0 \quad \forall x \neq 0, \tag{3.15}$$

*then $k(x)$ is a stabilizing feedback.*

*(b) Suppose $V$ is an RCLF for system (3.4). Then if $k(x)$ is such that for some $\alpha > 0$,*

$$\alpha^2 k^2(x) + \langle DV, F(x, k(x)) \rangle < 0 \quad \forall x \neq 0, \tag{3.16}$$

*then $k(x)$ is a robustly stabilizing feedback under the presence of (small enough) persistently acting disturbance $d(t)$, measurement errors $e(t)$, and external disturbances $w(t)$; i.e., $k(x)$ stabilizes system*

$$\dot{x} = F(x, k(x + e(t))) + G(x, k(x + e(t)))d(t) + w(t). \tag{3.17}$$

This lemma is similar to those smooth versions concerning stabilization and robust stabilization (see e.g. [21]), but due to discontinuous feedback great care must be taken here. A *generalized solution* has been introduced in [50, 31]. Stability concepts and theorems have been established for generalized solutions. We recap some concepts and results relevant to this thesis.

Let $\pi = \{t_i\}$ be a partition of the interval $[0, +\infty)$. A solution (called $\pi$-trajectory) is defined recursively on each $[t_i, t_{i+1}]$: at $t_i$, the state is measured, and correspondingly $u = k(x(t_i))$ is obtained and maintained until $t_{i+1}$. If on each $[t_i, t_{i+1}]$ the solution exists, then the solution is well defined for all time. By the diameter of the partition $\pi$, we mean $\mathrm{diam}(\pi) = \sup_{i \geq 0}(t_{i+1} - t_i)$.

**Definition 3.10.** *A feedback $k(x)$ is said to stabilize (3.4) if for each pair $(r, R)$, $0 < r < R$, there exist $M = M(R) > 0$, $\delta = \delta(r, R) > 0$, and $\tau = \tau(r, R) > 0$ such that, for every partition $\pi$ with $\mathrm{diam}(\pi) < \delta$ and for any initial state $x_0$ such that $|x_0| \leq R$ (where $|\cdot|$ is the Euclidean norm), the $\pi$-trajectory $x(\cdot)$ of the closed-loop system starting from $x_0$ is well defined, and it holds that:*

*(1) (uniform attractiveness) $|x(t)| \leq r \quad \forall t \geq \tau$;*

*(2) (overshoot boundedness) $|x(t)| \leq M(R) \quad \forall t \geq 0$;*

*(3) (Lyapunov stability) $\lim_{R \to 0} M(R) = 0$.*

*If under the presence of (small enough) persistently acting disturbances $d(t)$, measurement errors $e(t)$ with $|e(t)| \leq \phi$ for all $t \geq 0$, and external disturbances $w(t)$ with $\|w(\cdot)\|_\infty \leq \phi$ (where $\|\cdot\|_\infty$ is the norm in $L^\infty([0, T], \mathbb{R}^n)$), $k(x)$ is a stabilizing feedback for the closed-loop system*

$$\dot{x} = F_d(x, k(x + e(t)), d(t)) + w(t), \tag{3.18}$$

*then $k(x)$ is a robust stabilizing feedback for system (3.8).*

The following lemma is given in [30, 31]. It essentially links CLF's for (3.4) to stabilizing feedback, and UCLF's for (3.8) to robustly stabilizing feedback. Lemma 3.5 directly follows from this lemma.

**Lemma 3.6.** *(a) The control system (3.4) admits a CLF if and only if there exists a stabilizing feedback for it. Moreover, if a feedback $k(x)$ (either continuous or discontinuous) satisfies*

$$\langle DV, F(x, k(x)) \rangle < 0 \quad \forall x \neq 0,$$

*then $k$ is a stabilizing feedback for (3.4).*

*(b) The control system (3.8) admits a UCLF if and only if there exists a robustly stabilizing feedback for it. Moreover, if a feedback $k(x)$ (either continuous or discontinuous) satisfies*

$$\max_{d \in D} \langle DV, F_d(x, k(x), d) \rangle < 0 \quad \forall x \neq 0,$$

*then $k$ is a robustly stabilizing feedback for (3.8).*

The notion of *practical stabilizability* is given as follows.

**Definition 3.11.** *System (3.4) is said to be practically stabilizable, if there exists a CLF $V(x)$ such that, for any compact set $\mathcal{C}$ containing a neighborhood of the origin, and any $C_s = \{x \in X | V(x) \leq c_s\} \in \mathcal{C}$, there is a state-feedback controller $k(x)$, function of $\mathcal{C}$ and $c_s$, such that $\langle DV, F(x, k(x)) \rangle < 0$ for all $x \in \mathcal{C} \setminus C_s$, and such that any state in $C_s$ remains in it.*

By this definition, $C_s$ is an attractor of $\mathcal{C}$. Trajectories starting in $\mathcal{C}$ and outside $C_s$ will be attracted toward $C_s$ and will eventually enter it after finite time, and those starting in $\mathcal{C}$ never leave it.

## 3.4 Summary

In this chapter we have provided some useful mathematical preliminaries about quantized control systems. First, various concepts of quantizers have been introduced. The

SLQ is the key notion here. Second, the Lyapunov-based approach has been described. RCLF's and their properties are our major interest, and will be used to derive our main results. Third, the notions of stability and robustness for a quantized system have been presented.

# 4 PROBLEM DEFINITION AND MAIN RESULTS

In this chapter, we precisely define the problems we want to solve in this thesis and provide the solutions. We are interested in the following questions. What conditions imply the existence of quantized (robustly) stabilizing feedback of system (3.1)? What is the smallest possible density of the (robustly) stabilizing quantizers? How do we construct such a quantizer?

The first problem we are interested in is to find sufficient conditions that ensure the existence of (robustly) stabilizing quantizers. The solution to this problem has obvious theoretical significance.

**Problem 4.1.** *Find sufficient conditions that guarantee the existence of (robustly) stabilizing quantizers for system (3.1).*

The solution is given in the following theorem, in which (a) is applicable for the general nonlinear system (3.4).

**Theorem 4.1.** *(a) If $V$ is a CLF for system (3.4) on $S$, then (3.4) can be stabilized by a quantizer with a countable number of cells, or practically stabilized by a finite quantizer.*

*(b) If $V$ is an RCLF for system (3.1) on $S$, then (3.1) can be robustly stabilized by a finite density quantizer, or stabilized by a finite quantizer.*

Below, we outline the proof of (a), and leave the proof of (b) for later chapters.

*Proof.* Since $V$ is a CLF, for any $x_0 \neq 0$, there exists some $u_{x_0} \in \mathbb{R}$ such that

$$\left.\frac{\partial V}{\partial x}\right|_{x_0} F(x_0, u_{x_0}) < 0.$$

Then for the fixed $u_{x_0}$, we know $\frac{\partial V}{\partial x}F(x, u_{x_0})$ is continuous in $x$. Therefore, we can find some ball with radius $r_{x_0}$ centered in $x_0$, denoted $B(x_0, r_{x_0})$, such that if $x \in B(x_0, r_{x_0})$, then

$$\frac{\partial V}{\partial x}F(x, u_{x_0}) < 0.$$

By doing these procedures for all $x_0 \neq 0$ in $S$, we can get an open cover of $S \setminus \{0\}$ which is in a separable space $\mathbb{R}^n$. Thus, by the Lindelof Theorem (if a subset in a separable space has an open cover, then it has a countable subcover) [56], $S \setminus \{0\}$ has a countable subcover; i.e., there exist $x_{0_1}, x_{0_2}, \cdots$ such that

$$S \setminus \{0\} \subseteq \cup_{i=1}^{\infty} B(x_{0_i}, r_{x_{0_i}}).$$

Note that based on the countable subcover, we can obtain a partition of $S \setminus \{0\}$ (by eliminating overlap), and hence a quantizer with a countable number of cells. So (3.4) can be stabilized by a countable number of piecewise constant control inputs since they make $V$ decrease strictly.

Next we prove that for any given compact set $C$ containing the origin, trajectories starting in $S$ and outside $C$ will be attracted toward $C$, and those in $C$ will never leave it. This proof is simply done by following the procedures described above for all $x_0 \in K \triangleq \mathrm{cl}(S \setminus C)$, which forms an open cover for the compact set $K$. Then we can find a finite subcover of $K$. Based on the finite subcover, we can obtain a partition of $K$, and hence a quantizer with a finite number of cells on $K$. This completes the proof. □

By Lemma 3.1, we can easily establish the following corollary:

**Corollary 4.1.** *If a control-affine system (3.1) can be stabilized by a regular feedback, then it can be stabilized by a quantized feedback.*

The next problem we want to solve is finding a robustly stabilizing quantizer for system (3.1) if an RCLP is given. This is relatively easy to solve and the solution provides insight to the construction of quantizers if a CLF, instead of an RCLF, is available. More precisely, the problem is:

**Problem 4.2.** *Given an RCLP $(V(x), \alpha)$ for system (3.1) on $S$, $\alpha^2 < \alpha_M^2$, construct explicitly a quantizer $q \in \mathcal{Q}_\alpha(V)$, and, if possible, characterize the coarsest quantizer.*

We present the smallest density of such a robustly stabilizing quantizer in the following theorem. We can show that the coarsest one is *essentially semi-logarithmic*. We leave the details of the characterization of the coarsest one and the construction of robustly stabilizing quantizers for Chapter 5.

**Theorem 4.2.** *Suppose $(V(x), \alpha)$ is an RCLP for system (3.1) on $S$. If $\alpha_S^2 < +\infty$, then the coarsest quantizer $q^*$ for $(V(x), \alpha)$ has density $\eta^* = \frac{1}{\ln \frac{1}{\rho^*}}$, where $\rho^* = k_1/k_2$, $k_1 = \frac{-1+\sqrt{1-\alpha^2/\alpha_S^2}}{2\alpha^2}$, and $k_2 = \frac{-1-\sqrt{1-\alpha^2/\alpha_S^2}}{2\alpha^2}$. If $\alpha_S^2 = +\infty$, then the coarsest quantizer $q^*$ for $(V(x), \alpha)$ has density $\eta^* = 0$.*

The third problem we want to address is the construction of a stabilizing quantizer for system (3.1) if a CLF is given.

**Problem 4.3.** *Given a CLF $V(x)$ for system (3.1) on $S$, construct explicitly a quantizer $q$ such that for any $x \in S$, $x \neq 0$,*

$$L_g V(x) q(x) + L_f V(x) < 0, \tag{4.1}$$

*and, if possible, find the smallest density of stabilizing quantizers.*

The solution to this problem is called a *hierarchical semi-logarithmic quantizer* (HSLQ), which will be introduced and elaborated in Chapter 6. We show that the smallest density that guarantees stabilization is zero.

# 5 QUANTIZED ROBUST STABILIZATION BASED ON AN RCLF

In this chapter we present the solution to Problem 4.2. For a control-affine system with an RCLP given, we provide the smallest density of the robustly stabilizing quantizers, characterize such a quantizer, and present a way to construct robustly stabilizing quantizers. We show that several classes of control-affine systems, such as linear systems, feedback linearizable systems, and locally linearizable systems, fall into this category, and their special features are used to derive more specific quantization results.

## 5.1 Coarsest Quantization

In this section, we are interested in finding the smallest possible density of the robustly stabilizing quantizers. To avoid the trivial case that $u = 0$ can decrease the RCLF $V$ (i.e., $L_f V(x) < 0$ for all $x \neq 0$ in some neighborhood of the origin), we assume that in any neighborhood of the origin, there is some $x \neq 0$ such that $L_f V(x) > 0$. For the reader's convenience, we state the result (Theorem 4.2) again:

**Theorem 5.1.** *Suppose $(V(x), \alpha)$ is an RCLP for system (3.1) on $S$. If $\alpha_S^2 < +\infty$, then the coarsest quantizer $q^*$ for $(V(x), \alpha)$ has density $\eta^* = \frac{1}{\ln \frac{1}{\rho^*}}$, where $\rho^* = k_1/k_2$, $k_1 = \frac{-1+\sqrt{1-\alpha^2/\alpha_S^2}}{2\alpha^2}$, and $k_2 = \frac{-1-\sqrt{1-\alpha^2/\alpha_S^2}}{2\alpha^2}$. If $\alpha_S^2 = +\infty$, then the coarsest quantizer $q^*$ for $(V(x), \alpha)$ has density $\eta^* = 0$.*

The proof of this theorem (as well as the proofs of the lemmas and propositions in this section) is given in Appendix A. Note that the smaller the $\alpha^2$, the coarser the quantization is, but the less robust the closed-loop system is. Although this theorem gives the coarsest quantizer under some conditions, we want to remark that for a continuous-time system, quantization density is only a partial measure of the complexity of the interaction between the quantizer and the system dynamics, in contrast to discrete-time systems [16]. Other quantities related to information transmission/processing, such as the mean number of switchings per unit time, are also important [36].

We want to stress that $\eta^*$ provided in Theorem 5.1 may not be an achieved infimum over all robustly stabilizing quantizers for the given RCLP. In other words, in some cases $q^* \notin \mathcal{Q}_\alpha(V)$. The following lemma shows when the density $\eta^*$ is achievable.

**Lemma 5.1.** *$q^* \in \mathcal{Q}_\alpha(V)$ if and only if there is some $a > 0$ such that $\alpha_S^2 \leq h(x)$ for all $x$ in $S_{f_a} = \{x \in S_f | V(x) \leq a\}$.*

Next, we will construct the coarsest quantizer $q^*$ if $\eta^*$ is achievable, and in case $\eta^*$ is not achievable, we will construct a quantizer $q_\epsilon$ with density $\eta^* + \epsilon$ for any given $\epsilon > 0$. We will show that these quantizers are in fact *essentially semi-logarithmic quantizers.*

Notice that if $\Omega_Z \triangleq \{x \in S | L_f V(x) < -\delta \|x\|^2, \delta > 0\}$ is nonempty, then we only need to define quantizers on $\Omega_{NZ} = S \setminus \Omega_Z$ since on $\Omega_Z$ we can use zero control input to hold (3.14).

**Construction of the coarsest quantizer $q^*$**

Consider the case that $\eta^*$ is achievable. Suppose there is some $a > 0$ such that $\alpha_S^2 \leq h(x)$ for all $x$ in $S_{f_a}$. Let $S_a = \{x \in S | V(x) \leq a\}$, i.e., the smallest invariant set of $V(x)$ containing $S_{f_a}$. Then we can use a finite number of control values to drive the state from $S \setminus S_a$ into $S_a$ (see Theorem 4.1(b)), and then focus on a smaller invariant set $S_a$, on which the coarsest quantization turns out to be semi-logarithmic. Since a finite number of control values do not affect the quantizer density, we know the density is determined by the quantization defined on $S_a$.

**Proposition 5.1.** *Suppose $(V(x), \alpha)$ is an RCLP for system (3.1) on $S$, and there is some $a > 0$ such that $\alpha_S^2 \leq h(x)$ for all $x$ in $S_{f_a}$. Then there exists a robustly stabilizing quantizer $q^* \in \mathcal{Q}_\alpha(V)$ with density $\eta^*$. $q^*$ is an ESLQ: it has a finite number of cells on $S \setminus S_a$ which drive the state from $S \setminus S_a$ into $S_a$, and is $\rho^*$-based semi-logarithmic on $S_a \cap \Omega_{NZ}$ with $p(x) = L_g V(x)$ and $u_0 = k_1 \gamma_0$.*

**Construction of quantizer $q_\epsilon$**

Now we consider the case that $\eta^*$ may not be achievable. Any density $\eta$ such that $\eta > \eta^*$ is achievable; i.e., there exists a quantizer $q_\epsilon \in \mathcal{Q}_\alpha(V)$ with density $\eta_\epsilon = \eta^* + \epsilon$ for any $\epsilon > 0$. Let $\rho_\epsilon = e^{-\frac{1}{\eta_\epsilon}}$, $\alpha_\epsilon^2 = \frac{\alpha^2}{1-(\frac{1-\rho_\epsilon}{1+\rho_\epsilon})^2}$, $S_{f\epsilon} = \{x \in S_f | h(x) > \alpha_\epsilon^2\}$, and $S_\epsilon$ be the smallest closed invariant set of $V(x)$ containing $S_{f\epsilon}$. Then we can use a finite number of control values to drive the state from $S \setminus S_\epsilon$ into $S_\epsilon$, and on $S_\epsilon$ we can construct a robustly stabilizing quantizer with density $\eta_\epsilon$.

**Proposition 5.2.** *Suppose $(V(x), \alpha)$ is an RCLP for system (3.1) on $S$. Then for any given $\epsilon > 0$, there exists a robustly stabilizing quantizer $q_\epsilon \in \mathcal{Q}_\alpha(V)$ with density $\eta_\epsilon = \eta^* + \epsilon$. $q_\epsilon$ is an ESLQ: it has a finite number of cells on $S \setminus S_\epsilon$ which drive the state from $S \setminus S_\epsilon$ into $S_\epsilon$, and is $\rho_\epsilon$-based semi-logarithmic on $S_\epsilon \cap \Omega_{NZ}$ with $p(x) = L_g V(x)$ and $u_0 = k_{1\epsilon}\gamma_0$ where $k_{1\epsilon} = \frac{-1+\sqrt{1-\alpha^2/\alpha_\epsilon^2}}{2\alpha^2}$.*

This proposition can be used for a more general purpose: the construction of a robustly stabilizing quantizer with any given achievable density, since $\epsilon$ here is not necessarily small. Note the explicit construction of such a quantizer relies on solving $\alpha_S^2$ and $\alpha_M^2$ explicitly, which may be difficult in general. For several special classes of control-affine systems, such as linear systems, they can be easily computed, as we will show later. At any rate, the above theorem captures the fundamental law of robustly stabilizing quantization and has theoretic significance.

The SLQ that robustly stabilizes system (3.1) can be seen as a 2-level hierarchical quantizer, if we view the dividing of $S$ into $\Omega_{NZ}$ and $\Omega_Z$ as Level 0 partition, which is to be refined by the semi-log partition (called Level 1 partition). (Here we consider only an SLQ instead of an ESLQ since we can always take a smaller invariant set as $S$ on which an SLQ is built.) The 2-level hierarchical quantizer in the closed-loop can also be seen as a 2-level hierarchical automaton. For the 2-level HSLQ, the state-space partition and its associated automaton are illustrated in Figure 5.1.

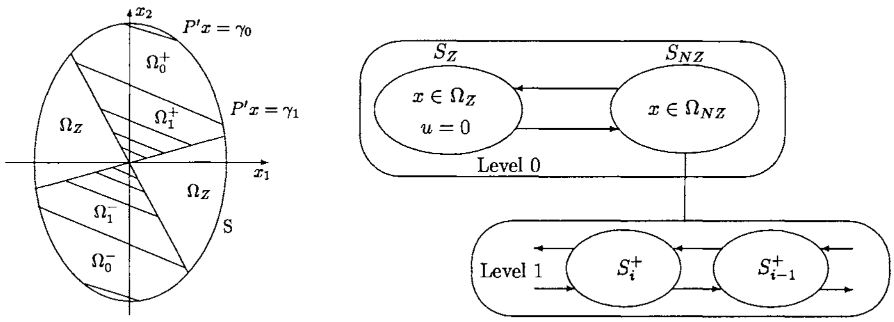


Figure 5.1 A 2-level hierarchical log quantization in the state-space and its associated 2-level hierarchical automaton. $S_Z$ denotes the state of the automaton associated with cell $\Omega_Z$, and $S_{NZ}$ for $\Omega_{NZ}$. The state $S_{NZ}$ in Level 0 is the automaton in Level 1.

## 5.2 Finite Quantizers

In practical circumstances, we are interested in finite quantizers. Infinite quantizers may not be applicable in practice since it requires infinite precision when the state approaches the origin. The next proposition shows that by truncating the countable infinite quantizer to a finite one, system (3.1) is still stabilized.

**Proposition 5.3.** *Suppose $q$ is an ESLQ on $S$ that robustly stabilizes system (3.1). Then its finite truncation on $S$ stabilizes (3.1) to the origin.*

*Proof.* Consider an ESLQ on $S$ that robustly stabilizes system (3.1). For some $j \in \mathbb{Z}$, let $\Omega_*^+ = \{x \in S | 0 < L_gV(x) \leq \gamma_j\}$, and $\Omega_*^- = \{x \in S | 0 > L_gV(x) \geq -\gamma_j\}$, and use $u^* = k_1\gamma_j$ in $\Omega_*^+$, $-u^*$ in $\Omega_*^-$. For any $i < j$, let $\Omega_i^\pm$ and the corresponding $u$ be as in Definition 3.3. Then we have a finite truncation of the robustly stabilizing quantizer, called $q_t$.

To show $q_t$ is stabilizing, we only have to show that on the compact set $S$, $x \neq 0$ implies $\dot{V} < 0$. In $\Omega_j^+$, since $0 < L_gV(x) \leq \gamma_j$ and $k_1 < 0$, we have

$$\dot{V} = L_fV + L_gVu^* = L_fV + L_gVk_1\gamma_j \leq L_fV + k_1(L_gV)^2. \tag{5.1}$$

If $L_fV \leq 0$, then the RHS of (5.1) is negative. If $L_fV > 0$, then direct calculation can show that

$$\frac{1}{4} + k_1 \frac{(L_gV)^2}{4L_fV} < 0.$$

So

$$L_fV + k_1(L_gV)^2 = 4L_fV(\frac{1}{4} + k_1 \frac{(L_gV)^2}{4L_fV}) < 0.$$

Therefore $q_t$ is a stabilizing quantizer. □

## 5.3 Quantization of Linear Systems

Linear systems are control-affine systems; hence the above results are applicable to linear systems. Moreover, we can show that every quadratic CLF (QCLF) is an RCLF for a linear system, and we can simplify the relevant computation. So, we can characterize its quantization more thoroughly.

Consider a single-input linear system

$$\dot{x} = Ax + Bu. \tag{5.2}$$

Suppose (5.2) has all its eigenvalues unstable but stabilizable. Then for any $Q > 0$, there exists a unique $P > 0$ such that

$$PA + A'P - PBB'P = -Q.$$

We know $V = x'Px$ is a CLF. Then $L_gV = 2x'PB$, and $L_fV = x'(PA + A'P)x$.

The following lemma says that any QCLF for linear system (5.2) is an RCLF, and $\alpha_S^2 = \alpha_M^2$ has an analytic formula.

**Lemma 5.2.** *If $V(x)$ is a QCLF for system (5.2), then $(V(x), \alpha)$ is an RCLP for system (5.2) if $0 < \alpha < \alpha_M$, where*

$$\alpha_M^2 = \alpha_S^2 = \frac{1}{1 - \frac{1}{\sigma^2}} > 1$$

*with $\sigma^2 = B'PQ^{-1}PB > 1$.*

The proof of this lemma (as well as the proofs of the following two propositions) is given in Appendix B. Using this lemma we can obtain the complete solution to Problem 4.2 for linear systems. That is, system (5.2) can be robustly stabilized by an LQ $q$; given an RCLP $(V(x), \alpha)$ as in Lemma 5.2, $q$ can be constructed as in Proposition 5.1 and is the coarsest for the RCLP.

**Proposition 5.4.** *Suppose $(V, \alpha)$ is an RCLP for system (5.2), where $V = x'Px$ is a QCLF with $P > 0$ such that $A'P + PA - PBB'P = -Q$ for some $Q > 0$, and $0 < \alpha^2 < \alpha_M^2 = \alpha_S^2 = \frac{1}{1-\frac{1}{\sigma^2}}$ with $\sigma^2 = B'PQ^{-1}PB > 1$. System (5.2) can be robustly stabilized by an LQ $q^*$, which is the coarsest quantizer for $(V, \alpha)$. $q^*$ can be derived from the RCLP as in Proposition 5.1.*

Therefore, for any $0 < \alpha^2 < \alpha_M^2$, a quantized control law can be directly derived from

$$\hat{u}^{(1),(2)} = \frac{-1 \pm \sqrt{1 - \alpha^2(1 - \frac{1}{\sigma^2})}}{\alpha^2} B'Px, \tag{5.3}$$

which can be rewritten in the following equivalent form

$$\hat{u}^{(1),(2)} = \frac{-1 \pm \frac{1}{\sigma_\alpha}}{\alpha^2} B'Px \tag{5.4}$$

where $\sigma_\alpha^2 = B'PQ_\alpha^{-1}PB$, and $Q_\alpha = PBB'P - \alpha^2(A'P + PA)$.

In [13, 16], the coarsest quantizer over all QCLF's has been found. Likewise we can find the coarsest quantizer over all QCLF's w.r.t. a given robustness level $\alpha$. Note that we require the QCLF's be normalized over the given compact set $S$; i.e., $V_M = 1$ (refer to Section 3.2). Without loss of generality, we can assume $S = \{x | x'x \leq 1\}$. We say a robustness level $\alpha$ is achievable on $S$ if there is some QCLF $V$ such that $(V, \alpha)$ is an RCLP.

**Proposition 5.5.** *For a given robustness level $\alpha > 0$ and a compact set $S = \{x | x'x \leq 1\}$,* *the robustness level $\alpha$ is achievable if and only if the LMI problem*

$$\begin{aligned} \inf \quad & \gamma \\ s.t.\ & RA' + AR - \gamma BB' < 0 \\ & R \geq I \\ & \gamma < 1 \\ & \alpha^2 \gamma < 1 \end{aligned} \tag{5.5}$$

*is feasible. Furthermore, if $(R^*, \gamma^*)$ is the optimal solution for problem (5.5), then the QCLF that gives the coarsest quantizer over all QCLF's is $V = x'(R^*)^{-1}x$, and the corresponding base is $\rho^* = \frac{1-\sqrt{1-\alpha^2\gamma^*}}{1+\sqrt{1-\alpha^2\gamma^*}}$.*

## 5.4 Quantization of Feedback Linearizable Systems

An input-to-state feedback linearizable system can be transformed into

$$\dot{x} = Ax + B\beta^{-1}(x)(u - \gamma(x)) \tag{5.6}$$

where $x \in S$, $u \in \mathbb{R}$, $B \in \mathbb{R}^n$, $(A, B)$ is controllable, $\beta(x)$ is nonsingular for all $x \in S$, $\gamma(0) = 0$, and $\beta(x), \gamma(x)$ are smooth [27].

The following supporting lemma guarantees that system (5.6) admits an RCLF, and its RCLF's can be found easily by investigating its linearization.

**Lemma 5.3.** *(a) If $V(x) = x'Px$ is a QCLF for system (5.2), then $V(x)$ is an RCLF for system (5.6).*

*(b) Suppose $V(x) = x'Px$ is a QCLF for system (5.2), $V_h(x)$ has an higher order than $V(x)$, and $V_s(x) = V(x) + V_h(x)$ is a CLF for system (5.6). Then $V_s(x)$ is an RCLF for system (5.6).*

We proceed with the main results and postpone the proof of this lemma in Appendix C. This lemma helps us to establish the following proposition regarding the quantized robust stabilization of system (5.6).

**Proposition 5.6.** *System (5.6) can be robustly stabilized by an SLQ $q$. $q$ can be derived from an RCLF $V_s = x'Px + V_h(x)$, where $P$ is the positive definite solution of $A'P + PA - PBB'P = -Q$ with $Q > 0$, and $V_h(x)$ has an higher order than $x'Px$.*

System (5.6) has two terms involving nonlinearity: $\beta^{-1}(x)$, the multiplicative nonlinearity, and $\gamma(x)$, the additive nonlinearity. If $|\beta^{-1}(x)| = 1$, then the multiplicative nonlinearity does not affect the system; if $|\beta^{-1}(x)| > 1$, we regard it as "good nonlinearity" since it helps us to save control effort, whereas if $|\beta^{-1}(x)| < 1$ it is regarded as "bad nonlinearity". Similarly, if $\gamma(x) \perp L_g V_s(x)$, then it has no effect on $\dot{V}_s$; if $\gamma(x) L_g V_s(x) > 0$, then it helps to make $V_s(x)$ decrease and thus it is "good nonlinearity", whereas if $\gamma(x) L_g V_s(x) < 0$ it is regarded as "bad nonlinearity".

Rather than cancelling all nonlinearity as does the feedback linearization approach, we want to make use of good nonlinearity to save control effort. Our major control effort is focused on bad nonlinearity. This also has advantages over some approaches in robust control design since they may be too conservative.

The quantizer is designed as follows (for simplicity we only consider a quadratic RCLF here):

(1) On the set $\{x \in S \| \beta^{-1}(x)| \geq 1, \gamma(x) L_g V(x) \geq 0\}$, use $\alpha_M^2 = \inf \frac{x'PBB'Px}{x'(A'P+PA)x}$, and

$$\hat{u}^{(1),(2)} = \frac{-1 \pm \sqrt{1 - \frac{\alpha^2}{\alpha_M^2}}}{\alpha^2} B'Px. \tag{5.7}$$

(2) On the set $\{x \in S \| \beta^{-1}(x)| \geq 1, \gamma(x) L_g V(x) < 0\}$, use $\alpha_M^2 = \inf \frac{x'PBB'Px}{x'(A'P+PA)x - \gamma(x) L_g V(x)}$, and

$$\hat{u}^{(1),(2)} = \frac{-1 \pm \sqrt{1 - \frac{\alpha^2}{\alpha_M^2}}}{\alpha^2} B'Px. \tag{5.8}$$

(3) On the set $\{x \in S \| \beta^{-1}(x)| < 1, \gamma(x) L_g V(x) \geq 0\}$, use $\alpha_M^2 = \inf \frac{x'PBB'Px\beta^{-2}(x)}{x'(A'P+PA)x}$, and

$$\hat{u}^{(1),(2)} = \frac{-1 \pm \sqrt{1 - \frac{\alpha^2}{\alpha_M^2}}}{\alpha^2} \beta^{-1}(x) B'Px. \tag{5.9}$$

(4) On the set $\{x \in S \| \beta^{-1}(x)| < 1, \gamma(x) L_g V(x) < 0\}$, use $\alpha_M^2 = \inf \frac{x'PBB'Px\beta^{-2}(x)}{x'(A'P+PA)x - \gamma(x) L_g V(x)}$, and

$$\hat{u}^{(1),(2)} = \frac{-1 \pm \sqrt{1 - \frac{\alpha^2}{\alpha_M^2}}}{\alpha^2} \beta^{-1}(x) B'Px. \tag{5.10}$$

## 5.5 Quantization of Locally Linearizable Systems

Suppose system (3.1) is locally linearizable; that is, system (3.1) can be written as

$$\dot{x} = Ax + Bu + f_1(x) + g_1(x)u \tag{5.11}$$

where $(A, B)$ is the linearization, $A$ is unstable, $(A, B)$ is stabilizable, and $f_1(x)$, $g_1(x)u$ are higher order terms.

The following lemma guarantees that system (5.11) admits a local RCLF $V$ (meaning the existence of a compact set around the origin on which $V$ is an RCLF), and $V$ can be found easily by investigating its linearization.

**Lemma 5.4.** *If a quadratic function $V(x) = x'Px$ is a CLF for system (5.2), and $V_h(x)$ has an higher order than $V(x)$, then $V_s(x) = V(x) + V_h(x)$ is a local RCLF for system (5.11).*

*Proof.* Since $V$ is a QCLF for system (5.2), we know $V_s$ is a local CLF for system (5.11). Because $V$ is quadratic, by Lemma 3.4 we know $V_s$ is an RCLF. □

This lemma helps us to establish the following proposition regarding the quantized robust stabilization of system (5.11).

**Proposition 5.7.** *System (5.11) can be locally robustly stabilized by an SLQ $q$. $q$ can be derived from an RCLF $V = x'Px + V_h(x)$, where $P$ is the positive definite solution of $A'P + PA - PBB'P = -Q$ with $Q > 0$, and $V_h(x)$ has an higher order than $x'Px$.*

*Proof.* The proof is a direct result of Lemma 5.4. □

## 5.6 Summary

In this chapter we have presented results on the quantized robust stabilization of a single-input nonlinear affine system if it admits an RCLF. We have provided the smallest density $\eta^*$ for an RCLP and have constructed the coarsest quantizer $q^*$ if $\eta^*$ is achievable. We have also constructed quantizer $q_\epsilon$ with density $\eta^* + \epsilon$ for any $\epsilon > 0$ ($\epsilon$ not necessarily small). Furthermore, we have shown that the finite truncation of a robustly stabilizing quantizer ensures stability of the closed-loop system. Finally, we have considered several important classes of control-affine systems. We have shown how their special structures can be exploited to find RCLF's and to design quantizers.

# 6 QUANTIZED STABILIZATION BASED ON A CLF

In this chapter we present the solution to Problem 4.3. For a control-affine system with a CLF given, we provide one way to construct a stabilizing quantizer. The quantizer is an HSLQ, and the closed-loop system is hierarchical. The finite truncation of a stabilizing HSLQ leads to practical stability of the closed-loop system.

## 6.1 Hierarchical Quantizers

For the reader's convenience, we provide system (3.1) again:

$$\dot{x} = f(x) + g(x)u; \quad f(0) = 0$$

where $f$ and $g$ are $C^1$ functions, $x \in X \subseteq \mathbb{R}^n$, $X$ is the state-space, $u \in \mathbf{U} \subseteq \mathbb{R}$, $\mathbf{U}$ is the admissible control set.

When implementing results of Chapter 5, sometimes we can only have a very small $\alpha$ over set $S$, causing control value $u$ to be very large, although smaller control in some subset of $S$ is possible. An approach to overcome this deficiency is to partition $S$ into disjoint subsets, and on each subset we define a quantizer $q_m$, which allows us to impose larger $\alpha$ in some subsets and ensure smaller control. This leads to a hierarchical quantization structure. This also gives us a method to design stabilizing quantizers for system (3.1) with a CLF (possibly not robust). These quantizers have a hierarchical semi-logarithmic structure, which is universal to all single-input nonlinear-affine continuous-time systems with CLF's available.

Let $\Omega_Z = \{x \in S | L_f V(x) < -\delta\|x\|^2, \delta > 0\}$, and $\Omega_{NZ} = S \setminus \Omega_Z$. We first partition $\Omega_{NZ} \setminus \{0\}$, the set on which we need nonzero control, into disjoint subsets $\{K_m\}_{m=1}^{\infty}$ with $0 \notin \mathrm{cl}(K_m)$ for all $m$. Let $\alpha_{Mm}^2 = \inf_{x \in K_m \cap S_f} h(x)$. Obviously, we have $\alpha_{Mm}^2 > 0$ for all

$m$. Then we can define a semi-log quantizer $q_m$ on each $K_m$. If the state is in $K_m$, $q_m$ is employed. If the state is driven outside of $K_m$ into $K_{m+1}$, quantizer $q_{m+1}$ is switched to. Each $q_m$ makes $V$ decrease and finally sends the state to the origin. This leads to a hierarchical semi-logarithmic quantization structure. In this thesis we call such a quantizer a *hierarchical semi-logarithmic quantizer* (HSLQ). Figure 6.1 illustrates the state-space partition of an HSLQ.

**Proposition 6.1.** *Suppose $V$ is a CLF for system (3.1) on $S$. System (3.1) can be stabilized to the origin by an HSLQ. Level 1 quantization is a partition of $\Omega_{NZ} \setminus \{0\}$ by disjoint sets $\{K_m\}_{m=1}^{\infty}$ with $0 \notin \mathrm{cl}(K_m)$ for all $m$. Level 2 quantization is obtained by defining a $\rho_m$-based SLQ $q_m$ on each set $K_m$ with $p(x) = L_gV(x)$ and $u_0 = k_{1m}\gamma_0$, where $\rho_m = k_{1m}/k_{2m}$, $k_{1m} = \frac{-1+\sqrt{1-\alpha_m^2/\alpha_{Mm}^2}}{2\alpha_m^2}$, $k_{2m} = \frac{-1-\sqrt{1-\alpha_m^2/\alpha_{Mm}^2}}{2\alpha_m^2}$, and $0 < \alpha_m^2 < \alpha_{Mm}^2$.*

*Proof.* It is sufficient to show that $\alpha_{Mm}^2 > 0$ on each $\mathrm{cl}(K_m)$; the proposition follows since on each $K_m$, $V$ continues to decrease. On the compact set $\mathrm{cl}(K_k) \cap \{x | L_fV(x) \geq 0\}$, we have $L_gV \neq 0$ (otherwise $L_fV < 0$, a contradiction), and $L_fV \neq \infty$; so $\frac{(L_gV)^2}{4L_fV}$ is bounded below by some positive number. Therefore $\alpha_{Mm}^2 > 0$. □

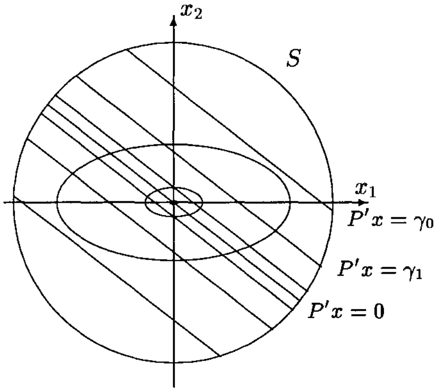


Figure 6.1 Example of Proposition 6.1: hierarchical log quantization of state-space. The ellipses show Level 1 quantization and the straight lines show Level 2 quantization. Level 2 quantization refines Level 1 quantization.

Proposition 6.1 implies that a general control architecture can be built for system (3.1) if (3.1) admits a CLF. This architecture is a 3-level hierarchical quantizer, if we call the partitioning of $S$ into $\Omega_{NZ}$ and $\Omega_Z$ *Level 0 partition.* Higher levels in the hierarchy manipulate only quantized information about the dynamics at lower levels. System (3.1) with the quantizer can be seen as a hierarchical automaton. The 3-level hierarchical automaton is illustrated in Figure 6.2. In this figure, if Level 1 quantization is decided by the level surfaces of $\|x\|$, then the logic conditions for state transition are: (1) $L_f V(x) \geq -\delta\|x\|^2$; (2) $L_f V(x) < -\delta\|x\|^2$; (3) $\|x\|^2 < c_m$; and (4) $\|x\|^2 \geq c_m$, where $\{c_m\}_{m=1}^{\infty}$ is a positive decreasing sequence.

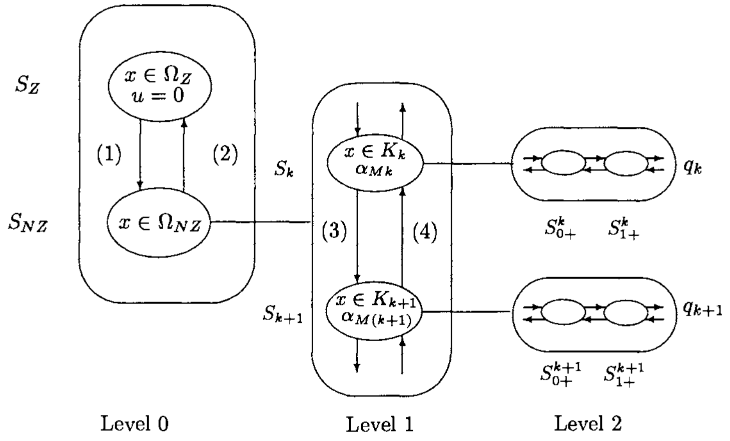


Figure 6.2 3-level hierarchical automaton for Proposition 6.1. The state $S_{NZ}$ in Level 0 is the automaton in Level 1. The state $S_k$ in Level 1 is the automaton $q_k$ in Level 2.

Note that here we require Level 1 partition to satisfy only the property $0 \notin \text{cl}(K_m)$. Designers have the freedom to design Level 1 partition. Normally Level 1 partition is given by the level surfaces of $V(x)$, $L_g V(x)$, $\|x\|$, etc. However, we do not have a way to optimize Level 1 and Level 2 partitions. Searching of the optimal joint partition will be the subject of future research.

Note that by assuming $V$ is a CLF for system (3.1) on $S$, we include the special case that $V$ is also an RCLF. The above result applies to a system with an RCLF. It is easy to see that if $V$ is an RCLF, then the HSLQ given by Proposition 6.1 ensures robust stabilization.

The above proposition helps us to establish the following result:

**Proposition 6.2.** *The smallest density of all quantizers that stabilize system (3.1) is zero.*

*Proof.* Consider the HSLQ designed in Proposition 6.1. Choose the sequence $\{\alpha_m\}_{m=1}^{\infty}$ such that $\frac{\alpha_{m+1}}{\alpha_{M(m+1)}} < \frac{\alpha_m}{2\alpha_{Mm}}$. Then $\rho_m$ converges to zero as $m$ goes to infinity. Consequently the density of the quantizer is zero. □

Although we have found the smallest density of all stabilizing quantizers, a quantizer with the smallest density may not have some "nice" properties such as finite gain, (see Section 6.3), and it leads to fast increase of the gain as the state approaches the origin. If we want the increase rate of the gain to be slow, we normally need to use infinite density quantizers. These quantizers may not be favored in practice. By relaxing the requirement of stabilization, we can construct quantizers with a finite number of cells and achieve practical stabilization, as we present in the next section.

## 6.2 Finite Hierarchical Quantizers

The HSLQ defined above has an infinite number of cells. The next proposition says that its finite truncation is practically stabilizing. The resulting finite quantizers are useful in practice.

**Proposition 6.3.** *Suppose $V$ is a smooth CLF of system (3.1) on $S$. System (3.1) can be practically stabilized to the origin by a finite hierarchical semi-logarithmic quantizer. The hierarchical quantization of the state-space is characterized as follows. Given an open set $N \subseteq S$, $0 \in N$, Level 1 quantization is defined as a finite partition of $S$ by disjoint sets $\{K_m\}_{m=1}^{l}$ and $N$, and Level 2 quantization is obtained by defining a $\rho_m$-based FSLQ $q_m$ on each set $K_m$ with $p(x) = L_gV(x)$ and $u_{0m} = k_{1m}\gamma_0$, where $\rho_m = k_{1m}/k_{2m}$, $k_{1m} = \frac{-1+\sqrt{1-\alpha_m^2/\alpha_{Mm}^2}}{2\alpha_m^2}$, and $k_{2m} = \frac{-1-\sqrt{1-\alpha_m^2/\alpha_{Mm}^2}}{2\alpha_m^2}$. On the set $N$, $u = 0$ is used.*

*Proof.* The proof requires only small modifications of previous ones. We skip it here. □

## 6.3 A Brief Discussion on CLF, RCLF, and Quantized Stabilization

So far, we have shown that, for the single-input nonlinear affine system (3.1), if a CLF is available, then it can be stabilized by an HSLQ with a countable number of cells, or practically stabilized by a finite HSLQ; if an RCLF is available, then it can be robustly stabilized by an ESLQ with a countable number of cells, or stabilized by an FSLQ. These quantizers are explicitly constructed; i.e., we have proven Theorem 4.1 in a constructive way.

It is clear from above that the availability of an RCLF guarantees some "nice" properties of quantization; that is, a simple partition (semi-log partition instead of hierarchical semi-log partition) is obtained, robust stabilization is achieved, and finite gain (from $x$ to $u$) is ensured. However, if the only available CLF is not robust, i.e., $\alpha_M^2 = \alpha_S^2 = 0$, the above mentioned properties may not be achieved. As the state approaches the origin, the quantizer constructed in Proposition 6.1 normally generates higher and higher gain, and leads to infinite gain at the origin. The increase rate of the gain as the state approaches the origin has a certain connection with the quantization density: a slower increase rate of the gain requires a larger density.

## 6.4 Summary

In this chapter we have presented results on the quantized stabilization of a single-input nonlinear affine system if it admits a CLF. The stabilizing quantizer designed here is hierarchical semi-logarithmic. By truncating this stabilizing quantizer with infinite cells, we have obtained a finite quantizer that guarantees practical stabilization. A by-product result is that the quantization constructed here results in a hierarchical control system.

# 7 FURTHER DISCUSSION

In this chapter we discuss briefly some interesting but unsolved topics relevant to the results we obtained, such as chattering-free quantizers, attention cost, and quantization of partially feedback linearizable systems. We do not intend to provide complete solutions to these problems since they are generally difficult to solve. This preliminary research lays a foundation for future work.

## 7.1 Chattering-free Quantizers

By *switching*, we mean the controller changing its control value. For a continuous-time system, when a quantized control law is employed, the RHS vector field is discontinuous, which may lead to *chattering*, or repeated infinitely fast switching. When such a quantized control law is implemented, it leads to fast switching (and not infinitely fast switching, since that is not possible for a physical system), which can be harmful, and requires more communication overhead. In this section we eliminate chattering in quantized control by applying *switching control with dwell time* developed and used in [8, 24]. In this approach, switching logic with a fixed dwell time is used so that only finite switchings can happen in finite time.

By *triggering*, we mean the state $x$ crossing the boundary of a cell; that is, in the $x$-$u$ space the state reaches the triggering manifold, and a triggering signal is sent. A switching logic with a fixed dwell time $T$ basically says that a triggering gives rise to a switching only if during $T$ before the switching occurs, there exists no other switching. The details are described as follows. When a switching occurs, the controller starts a timer. During the timing, no switchings are initiated; triggerings are ignored. When time interval $T$ has elapsed, the state is checked. If the state is found to be in a cell other than that which it entered at the switching, a new switching occurs. Else, the next triggering causes the next switching. Clearly, consecutive switching times have a minimum separation of $T$.

**Lemma 7.1.** *Suppose $q(x)$ is a stabilizing quantized feedback for (3.1) on $S$. Then for any $r > 0$ there exists some $T > 0$ ($T$ depending on $r$) such that $q(x)$ with dwell time $T$ practically stabilizes (3.1) to the $r$-ball of the origin. In addition, if*

$$T_M = \inf_{\substack{x \in S \\ L^2_{f+gq}V > 0}} \frac{W(x)}{L^2_{f+gq}V} > 0, \tag{7.1}$$

*where $W(x)$ is some positive definite function such that $L_fV(x) + L_gV(x)q(x) < -W(x)$ for all $x$ except 0, then any $0 < T \leq T_M$ is such that $q(x)$ with dwell time $T$ practically stabilizes (3.1).*

Applying this lemma to previous propositions and corollaries, we see that these quantizers can be made chattering-free and guarantee practical stability of the closed-loop systems.

*Proof.* Since $q(x)$ is stabilizing on $S$, for the given $r$ there is some $\delta > 0$ by Definition 3.10 such that for every partition $\pi$ with $\operatorname{diam}(\pi) < \delta$, the $\pi$-trajectory $x(\cdot)$ of the closed-loop system is well defined and satisfies (1)-(3) of Definition 3.10. Let $T < \delta$. We want to show that $q(x)$ with dwell time $T$ can practically stabilize (3.1) to the set $\{x||x| < r\}$ (where $|\cdot|$ is the Euclidean norm).

First we construct a partition $\pi_1 = \{t_i\}$ starting from $t = 0$ by letting $t_i$ be the $i$th switching time as we described above. Next we refine the partition $\pi_1$ by adding points in the interval $[t_i, t_{i+1}]$ for $i \geq 1$. If the length of the interval is shorter than $T$, we do not add points; if the length is larger than $T$, we add points to this interval until the distance between every two successive points is less than $T$. Finally we get a refined partition, denoted $\pi_2$, and $\operatorname{diam}(\pi_2) < \delta$. Therefore, the quantizer $q(x)$ with the partition $\pi_2$ can drive the state into the $r$-ball of the origin.

Observe that no triggering nor switching occurs at the added points. Hence we can remove these points from $\pi_2$ and the resulting partition still ensures that the state can be driven into the $r$-ball of the origin. Then we know $q(x)$ with the partition $\pi_1$ can practically stabilize (3.1). That is, the dwell time practically stabilizing quantizer exists.

Now we show the second part of the lemma. The idea is that by using the positive definite function $W(x)$ in (3.6), we can tolerate "imprecise" quantized input when switching logic with dwell time is used, and the resulting closed-loop system is practically stabilized. Here we say "imprecise" since the switching logic based on state-space partition is exactly followed only if no triggering occurs during $T$ after a switching. So, we want to find some $T$ small enough so that in the dwell time, $\dot{V}$ has only very small change, which can be tolerated by $W(x)$. During $T$, $\dot{V}$ may have a small variation $\Delta\dot{V}$, which can be approximated by $\Delta\dot{V} \approx \ddot{V}T = L^2_{f+gu}VT$. Note that during the dwell time $T$, no switching occurs, so $u$ is constant and its derivative is 0. If $\Delta\dot{V} \leq 0$, then $\dot{V} < 0$ is still valid. If $\Delta\dot{V} > 0$ but $\Delta\dot{V} \leq W(x)$ holds for some $T$ chosen to be sufficiently small, then we know the closed-loop system is still practically stable. That is, (7.1) is a sufficient condition. □

The drawback of this lemma is that it does not preserve robust stabilization, nor stabilization, as we have achieved before. The only result we can achieve using this lemma is practical stabilization. How this can be improved is still unknown and will be investigated in future research.

## 7.2 Comments on Attention

In this section we briefly discuss the relation between attention and our results about quantization. We show that our quantizers do reduce the attention cost; hence they may be useful in practice.

Attention is investigated in [7, 36, 16] and is interpreted as the effort of implementation of a control law, although in these papers the definitions of attention cost take different forms to accommodate different systems. Again, we need to redefine attention cost to focus on measuring the interaction between the controller and the plant. The attention functional introduced in [7] leads to zero cost for our quantized systems, since $u$ has first partial derivatives equal to zero almost everywhere. [36] defines attention cost as the mean value of switchings per unit time. However, to measure the required effort of the control law it might be necessary to include the length of data to be transmitted/processed. [16] links state-space$\times$time quantization density to attention cost for LQ's with infinite cells.

We would like to define attention cost for quantized control so that it captures the following intuitions: 1) attention cost takes into account communication effort (or, attention of information transmission through communication channels) and computation effort (or, attention of information processing); 2) faster switchings results in larger attention cost; and 3) longer data (also called symbol) results in larger attention cost.

The intuitions listed above suggest we define attention cost as $AC = \frac{1}{T}\lceil \log_2 N \rceil E_b$, where $T$ is the mean time between switchings ($\frac{1}{T}$ is the symbol rate), $N$ is the number of cells of the quantizer (a larger $N$ requires more bits to represent state information and control commands), $\lceil \log_2 N \rceil$ is the number of bits for one symbol assuming symbols are equiprobable (with $\lceil a \rceil$ giving the nearest integer no smaller than $a$), and $E_b$ is the energy spent for transmitting/processing one bit (which resembles "bit energy" in communication theory).

This definition of attention cost coincides with the concept of average power of digital signals in communication theory, except that $E_b$ here takes into account both communication and computation effort.

If a control law has infinite attention cost, then it either has infinite bits to transmit/process each time, or it must transmit/process data infinitely fast, which means the control law is difficult to implement using practical digital channels/computers. A control law with finite attention cost implies that there are only finite interactions in finite time (attention of the channel/controller is coarsely distributed along the time axis), and each time an interaction occurs, a finite-bit data-signal is handled (finite energy is consumed), which is suitable for practical use. Furthermore, the larger the attention cost, the more challenging is the communication/computation process. Therefore, we consider our definition of attention cost suitable for measuring communication/computation effort.

Traditional control schemes cost infinite attention since switchings are continuously distributed along the time axis and infinite precision of data is assumed. Countable infinite quantizers and quantizers with chattering also have infinite attention cost. Finite quantizers obtained in this thesis can be made chattering-free by applying Lemma 7.1. For these quantizers, we have the following result:

**Corollary 7.1.** *For finite quantizers without chattering, the attention cost is finite.*

*Proof.* Immediate and skipped. □

This corollary ensures that results on quantization in this thesis are indeed meaningful for practical applications.

Notice that attention cost can be reduced by paying attention only on demand. [16] investigates a uniform sampling strategy together with logarithmic quantizers for continuous LTI systems. When the logarithmic quantizer is truncated to be finite, it results in finite attention cost as defined in this thesis. Although uniform sampling is widely used and has obvious advantages, it may not lead to minimum attention cost since it fixes the time to pay attention *a priori*. In contrast, the quantizers built in this thesis allow us to pay attention in a more flexible way, thus increasing the possibility of obtaining less costly control. In fact, the quantizers designed in this thesis pay attention only when attention is needed. That is, attention (and communication/computation effort) is used on demand. Only when the decrease of the CLF is not fast enough, an interrupt is sent and attention is requested and given. In some examples we will show later, we observe that most of the time the quantized controllers do nothing but wait for the interrupt request coming. Thus such a quantized control law can be implemented without diverting much attention from other tasks which are more pressing, and the efficiency of communication/computation resources is improved.

We would like to point out that the attention cost can measure communication/computation effort more precisely if we study the coding scheme and calculate the mean length of one symbol. This requires study of the dynamics of resulting quantized systems (which is relevant to the work described in Section 2.3) and will be investigated in future work.

As a final comment, we remark that in practice attention cost should be considered together with other factors. For one thing, our definition of attention cost measures only the *average* effort of a control law; however, to allocate communication/computation resources efficiently we need to consider other characteristics such as maximum instantaneous effort. Another reason is that low attention cost may imply slow convergence of the states; a balance between the effort of the control law and the performance of the system should be considered, as is done in [7].

## 7.3 Quantization of Partially Feedback Linearizable Systems

A partially feedback linearizable system can be transformed into

$$\begin{aligned} \dot{\xi} &= A\xi + B\beta^{-1}(\xi,\eta)(u-\gamma(\xi,\eta)) \\ \dot{\eta} &= q(\xi,\eta) \end{aligned} \tag{7.2}$$

where $x = (\xi', \eta')' \in S \subseteq \mathbb{R}^n$, $\xi \in \mathbb{R}^r$, $\eta \in \mathbb{R}^{n-r}$, $u \in \mathbb{R}$, $B \in \mathbb{R}^r$, $(A, B)$ is controllable, $\beta(x)$ is nonsingular for all $x \in S$, $\gamma(0) = 0$, and $\beta(x), \gamma(x)$ are smooth. [25, 27]

Through an invertible change of control of the form

$$u = \beta(\xi,\eta)v + \gamma(\xi,\eta), \tag{7.3}$$

system (7.2) is partially linearized; that is

$$\begin{aligned} \dot{\xi} &= A\xi + Bv \\ \dot{\eta} &= q(\xi,\eta). \end{aligned} \tag{7.4}$$

The main result for partially feedback linearizable systems is that, if the partially linearized system (7.4) admits a quadratic RCLF $V(x)$, then $V(x)$ is also an RCLF for system (7.2), and semi-logarithmic quantization can be directly derived from $V(x)$.

Therefore, once a quadratic RCLF for system (7.4) is found, we can robustly stabilizing system (7.2) by an SLQ. The proofs of the results in this section are given in Appendix D.

**Proposition 7.1.** *If system (7.4) admits a quadratic RCLF* $V = x'Px$, *then system (7.2) can be robustly stabilized by an SLQ* $q_s$*;* $q_s$ *can be derived from the RCLF* $V$.

This proposition follows directly from the following lemma.

**Lemma 7.2.** *If a quadratic function* $V(x)$ *is an RCLF for system (7.4), then it is an RCLF for system (7.2).*

Therefore, the problem of quantizing system (7.2) is reduced to the problem of searching for a quadratic RCLF for system (7.4). If (7.4) has certain structure, e.g., strict-feedback structure ([27, 29]), then there are certain procedures that can be followed to find a QCLF, which sometimes can be shown to be an RCLF. Below, we provide another way to quantize (7.2) based on a quadratic RCLF constructed for (7.4) (and hence for (7.2)) under certain conditions. A general method is still missing.

**Corollary 7.2.** *For system (7.2), suppose* $P$ *is the positive definite solution of* $A'P + PA - PBB'P = -Q$ *with* $Q > 0$*, and* $W(\eta)$ *is a quadratic Lyapunov function for the zero dynamics* $\dot{\eta} = q(0, \eta)$ *with* $\frac{dW}{d\eta} q(0, \eta) < -c_1 \|\eta\|^2$ *for some* $c_1 > 0$*. Then there exists some* $c > 0$ *such that* $V = 2\xi' P \xi + cW(\eta)$ *is an RCLF for system (7.2), and system (7.2) can be robustly stabilized by an SLQ* $q_s$*;* $q_s$ *can be derived from the RCLF* $V$.

This corollary relies on the following supporting lemmas.

**Lemma 7.3.** $V(x)$ *is a CLF for system (7.2) if and only if* $V(x)$ *is a CLF for system (7.4).*

This lemma shows that to find a CLF for system (7.2) we need only to search for a CLF for system (7.4). The following lemma may be useful in constructing a CLF for system (7.4).

**Lemma 7.4.** *For system (7.4), suppose $P$ is the positive definite solution of $A'P+PA-PBB'P=-Q$ with $Q>0$, $W(\eta)$ is a Lyapunov function for the zero dynamics $\dot{\eta}=q(0,\eta)$, and there exists $c_1>0$ such that $\frac{dW}{d\eta}q(0,\eta)<-c_1\|\eta\|^2$. Then $V=\xi'P\xi+c_2W(\eta)$ is a CLF for system (7.4) for some $c_2>0$, and $v=-\frac{1}{2}B'P\xi$ is a stabilizing control.*

The following lemma constructs an RCLF for system (7.4).

**Lemma 7.5.** *For system (7.4), suppose $P$ is the positive definite solution of $A'P+PA-PBB'P=-Q$ with $Q>0$, and $V_1=\xi'P\xi+cW(\eta)$ is a CLF for system (7.4) for some $c>0$ and some function $W(\eta)$. Then $V=2\xi'P\xi+cW(\eta)$ is an RCLF for system (7.4).*

## 7.4 Summary

In this chapter we have discussed briefly some topics relevant to the results we obtained, such as chattering-free quantizers and attention cost. The quantizers designed before this chapter might not be possible for implementation, since they may cause chattering. We have discussed how to eliminate chattering by introducing dwell-time switching logic, and we have performed preliminary research on attention cost. We have also discussed the quantized control of partially feedback linearizable systems. These systems are much more complicated, and we derived some preliminary results regarding the existence of a quadratic RCLF. The limitations and deficiencies of these results were also discussed.

# 8 SIMULATION

In this chapter, we apply the results derived in preceding chapters to several control-affine systems. These include a unicycle-type vehicle model and a car-like vehicle model. We show that our quantized control law not only leads to desired stability results, but also reduces the interaction between the controller and the plant.

## 8.1 Quantized Control for a Unicycle-type Vehicle

In this section we design a quantized control law to steer a unicycle-type vehicle. We are interested in quantized control of autonomous vehicles and mobile robots (quantized motion control) for several reasons. First, autonomous vehicles are subject to computational complexity, communication constraints, and real-time requirements; and the quantization-based approach is an effective way to reduce the interaction between the controller and the system. Second, we would like to understand the intrinsic difficulty of controlling (steering) a vehicle in an obstacle-free space. We will see that quantized control leads to a natural behavior using a finite number of simple control primitives. While illustrating the idea of quantized control, we will not consider the issue of nonholonomy (many autonomous vehicles are subject to nonholonomic constraints; cf. [40, 28]); we simply remove nonholonomic constraints by setting some inputs constant [39].

Following [20], we use trim trajectories as the control primitives. Roughly speaking, for a unicycle-type vehicle in a plane, trim trajectories are obtained by steering the vehicle with constant linear velocities and angular velocities. The set of trim trajectories contains the straight line and all circle arcs with different radii. Each trim trajectory is corresponding to one constant control input, and all these constant control inputs form a set $\mathcal{T}$. By *Quantized motion control* we mean the admissible control inputs are restricted to an at most countable set $\mathcal{T}_c$, subset of $\mathcal{T}$. Between two switchings, the quantizer holds the control input, and this constant input steers the vehicle to move along some trim trajectory until at the next switching time the quantizer selects another control input from $\mathcal{T}_c$. In this paper $\mathcal{T}_c$ is generated by a given CLF. An additional advantage of the Lyapunov-based approach is that it does not require identification of the control primitives beforehand.

### 8.1.1 Model of a Unicycle-type Vehicle

The kinematics of a unicycle-type vehicle in a 2-D plane can be modeled as follows:

$$\begin{aligned} \dot{x_1} &= -u_1 \cos x_3 \\ \dot{x_2} &= u_1 \sin x_3 \\ \dot{x_3} &= u_2 \end{aligned} \tag{8.1}$$

where $x_3$ is the steering angle, $u_1$ is the linear velocity, and $u_2$ is the steering angular velocity. $1/u_2$ is the radius of the turn made by the vehicle. As mentioned before, we set $u_1$ constant to make the system holonomic. We denote $u = u_2$ and let $u_1 = 1$ without loss of generality. The above equations are rewritten as

$$\begin{aligned} \dot{x_1} &= -\cos x_3 \\ \dot{x_2} &= \sin x_3 \\ \dot{x_3} &= u. \end{aligned} \tag{8.2}$$

We focus on two tasks of motion control in an obstacle-free plane: 1) line tracking: the vehicle is required to go along a straight line; and 2) position control: the desired state is a fixed position in the plane.

### 8.1.2 Quantized Line Tracking

We consider the problem that system (8.2) is required to track the $x_1$ axis in the $x_1$-$x_2$ plane and point due west (left). (Note that we can always choose the desired path as the $x_1$ axis; therefore, the quantizer design method can be used to track *any* straight

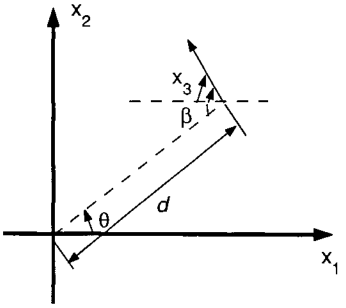


Figure 8.1 Kinematic model of a unicycle-type vehicle. $x_3 \in (-\pi, \pi]$ is the angle between heading direction and the $x_1$ axis (west), $d^2 = x_1^2 + x_2^2$, $\theta = \text{atan2}(x_2, x_1)$, and $\beta = x_3 + \theta\ (\pm 2\pi) \in (-\pi, \pi]$.

line in the plane.) Since there is no requirement on $x_1$, we focus only on the dynamics of $x_2$ and $x_3$:

$$\begin{aligned} \dot{x_2} &= \sin x_3 \\ \dot{x_3} &= u. \end{aligned} \tag{8.3}$$

Once $x_2$ and $x_3$ are stabilized, the vehicle will be running along the desired trajectory.

System (8.3) is locally linearizable and the linearization is stabilizable, so it admits an RCLF. It is easily verified that $V = x_2^2 + x_3^2 + x_2 x_3$ is an RCLF for the dynamics of $x_2$ and $x_3$. One way to verify this is to apply Lemma 3.3, which gives us $\alpha_M^2 = \alpha_S^2 = 3/4$. The other way is to first show that $V$ is a CLF for the linearized system, and then show $V$ is a CLF for system (8.3). (This can be done easily since we only need to verify $L_g V = 0$ implies $L_f V < 0$ if $x \neq 0$).

We can furthermore verify that in any neighborhood of the origin, there is some $(x_2, x_3)$ such that $L_f V(x) > 0$. Thus Propositions 5.1 and 5.3 are applicable.

**Quantizer Design 8.1.** *System (8.3) can be robustly stabilized by an LQ. Level 0 partition is given by* $\Omega_Z = \{x | \sin x_3(2x_2 + x_3) < -\delta(x_2^2 + x_3^2)\}$ *and* $\Omega_{NZ} = \{x | \sin x_3(2x_2 + x_3) \geq -\delta(x_2^2 + x_3^2)\}$ *for some* $\delta > 0$. $\Omega_{NZ}$ *is logarithmically partitioned by* $p(x) = x_2 + 2x_3$ *(Level 1 partition). Finite truncation of this quantizer is stabilizing.*

Figure 8.2 is a sample trajectory using the quantizer given above. The vehicle is running in the $x_1$-$x_2$ plane. A 2-level quantization is defined on the $x_2$-$x_3$ plane. The dashed line is the desired trajectory in the $x_1$-$x_2$ plane. The trajectory has been plotted in the $x_1$-$x_2$-$x_3$ space, as well as projections in the $x_1$-$x_2$ and $x_2$-$x_3$ planes. Stars represent the switching points. We can see from the figure that the vehicle follows a natural trajectory to reach the desired trajectory and then goes along it. Interaction between the quantizer and the vehicle exists only at the star points.

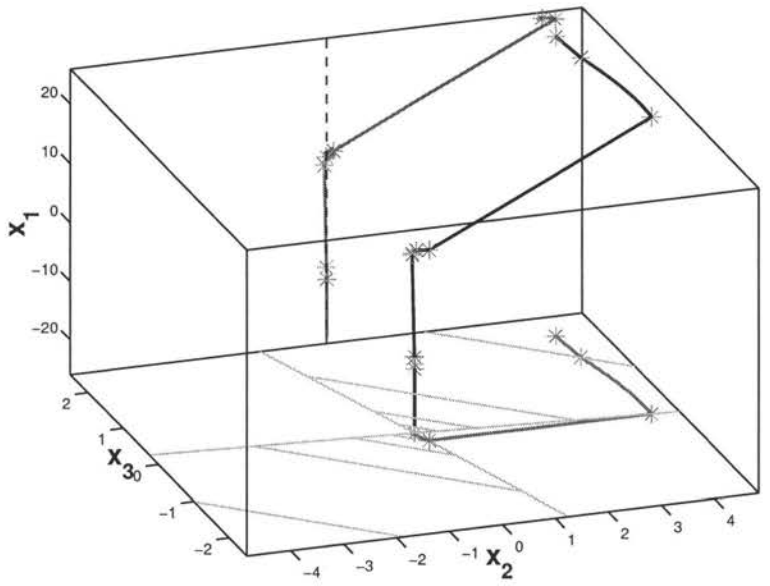


Figure 8.2 A sample trajectory of the vehicle's line tracking.

In Figure 8.3, a comparison is made between two quantizers with different densities. In **A**, a denser quantizer is used, and 5 bits are needed to represent the necessary control primitives that steer the vehicle to the desired trajectory with high precision, whereas in **C** only 3 bits are used. We find that **A** has faster switchings than **C**. Thus A has more interaction between the quantizer and the vehicle. We also observe a smaller overshoot and faster convergence in **A**. There is a trade-off between better performance and less interaction. So, in practical use, it is important to choose a suitable quantizer to achieve the desired control specifications while using minimal control effort. This may be formulated as the problem of "minimum attention quantization" that attempts to find the balance between performance and the difficulty involved in implementing the control, as introduced in [7].

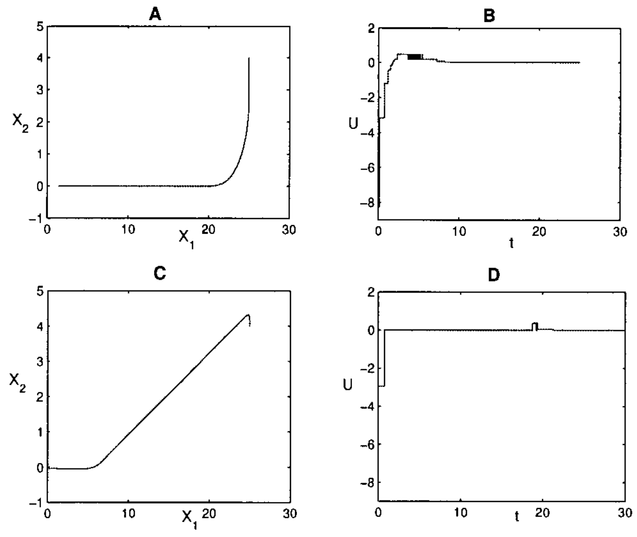


Figure 8.3 Sample trajectories and control signals for the line tracking of a unicycle. **A** is the trajectory using a denser quantizer, and **B** shows its input; **C** is the trajectory using a coarser quantizer with the same initial conditions, and **D** shows its input.

### 8.1.3 Quantized Position Control

In this subsection we consider the problem that the origin of the $x_1$-$x_2$ plane is the desired destination. Now we need to drive $x_1$ and $x_2$ to zero, and there is no requirement on $x_3$. Since the origin of the $x_1$-$x_2$ plane is no longer an equilibrium point of the system when we assume constant linear velocity, there exists no real CLF. To find a Lyapunov-like function, which is a CLF for part of the dynamics, it is helpful to change (8.2) to the following form [1]:

$$\begin{aligned} \dot{d} &= -\cos\beta \\ \dot{\beta} &= \tfrac{1}{d}\sin\beta + u \end{aligned} \tag{8.4}$$

where $d = \sqrt{x_1^2 + x_2^2}$ is the distance between the vehicle and the destination, and $\beta = x_3 + \theta\ (\pm 2\pi) \in (-\pi, \pi]$ is the deviation angle between the heading direction and the destination, where $\theta = \text{atan2}(x_2, x_1)$. The dynamics $\dot{x}_3 = u$ does not appear in (8.4) since it does not influence the dynamics of $d$ and $\beta$ and we do not have any requirement on $x_3$. Let us assume at the start $d \neq 0$. Now we can find a CLF for the second equation in (8.4), and $\beta$ can be driven to zero if $d \neq 0$. So we can apply the results of Chapter 6 to design a quantizer. To illustrate how to design a finite quantizer, we use Proposition 6.3 here.

**Quantizer Design 8.2.** *Given any $d^* > 0$. The trajectory of system (8.4) will converge to the set $D^* = \{d | d < d^*\}$ by applying a hierarchical quantizer. Let $k$ be any integer, $s > 2$, $\rho = s - 1 - \sqrt{s(s-2)} < 1$, $D_1 = \{d | d \geq d^*/\rho^{k-1}\}$, and $D_i = \{d | d^*/\rho^{k-i+1} > d \geq d^*/\rho^{k-i}\}$ for $i = 2, 3, \ldots, k$. Level 1 quantization is a finite $\rho-$based logarithmic partition along the $d$ axis by disjoint sets $\{D_i\}_{i=1}^k$ and $D^*$. Level 2 quantization is given by defining a finite $\rho-$based logarithmic quantizer $q_i$ on each $D_i$ by partitioning $D_i$ along the $\beta$ axis, and using $\alpha_{Mi}^2 = \frac{d^*}{2\rho^{k-i}}$ and $\alpha_i^2 = 2\alpha_{Mi}^2/s$ for quantizer $q_i$.*

The generation of the control input by quantized information of $d$ and $\beta$ is illustrated in Figure 8.4. In this figure, $\mathbf{1}_{D_i}(d)$ denotes the indicator function: if $d \in D_i$ then it takes value 1; otherwise it takes value 0. Since we let all partitions have the same base $\rho$, we can reduce the data to be stored and simplify calculation. (We only need to store $d^*$, $k$, and $s$ in memory, and all other quantities can be computed through simple computation; here $u = q_k(j) = \frac{-1+\sqrt{1-2/s}}{\alpha_k^2}\rho^j$.)

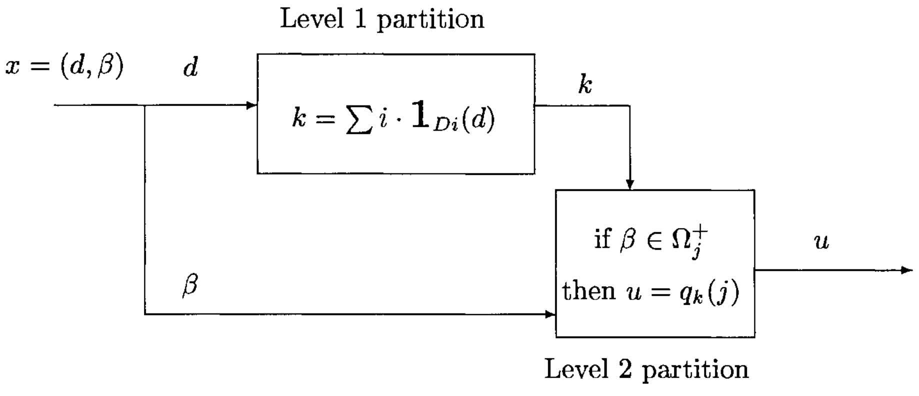


Figure 8.4 The hierarchical structure to generate $u$ based on quantized information of $d$ and $\beta$.

The justification of this quantizer design is simple. Let $V = \beta^2$. Then on the compact set $K = \{(d, \beta) | d \in [d^*/\rho^{k-i}, d_M], \beta \in [-\pi, \pi]\}$ for any $d_M > d^*/\rho^{k-i}$, we have

$\alpha_M^2 = \frac{d^*}{2\rho^{k-i}}$. The result follows.

An interesting result is that by using quantization we can simultaneously control more than one vehicle using one quantized controller. Fig 8.5 shows sample trajectories for the motion of three unicycles. The initial headings are all due north. The figure also shows the quantized control signals. During this process, interaction between the quantizer and a vehicle is coarsely distributed along the time axis, and the probability that multiple vehicles need interaction at exactly the same time is very small (if the vehicles are not started at the same time).

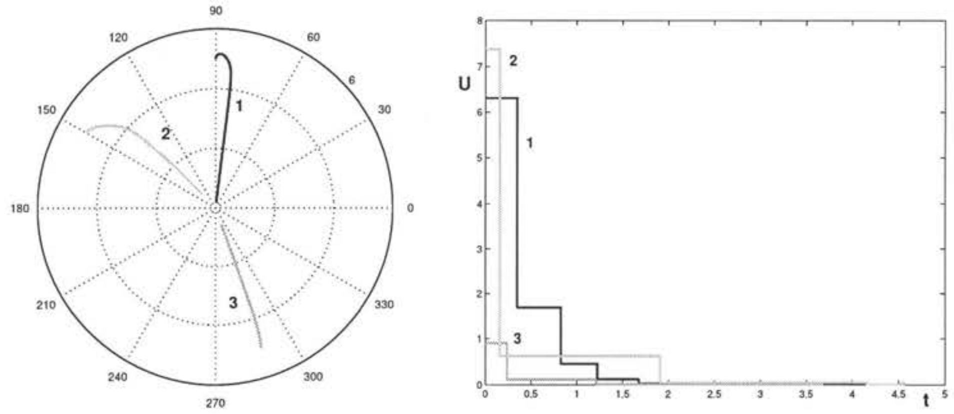


Figure 8.5 Sample trajectories and control signals for the position control of three unicycles.

## 8.2 Quantized Control for a Car-like Vehicle

### 8.2.1 Model of a Car-like Vehicle

The kinematics of a car-like vehicle are modeled as follows:

$$\begin{aligned} \dot{x_1} &= u_1 \cos x_3 \\ \dot{x_2} &= u_1 \sin x_3 \\ \dot{x_3} &= \tfrac{1}{l} u_1 \tan \phi \\ \dot{\phi} &= u_2 \end{aligned} \tag{8.5}$$

where $x_1, x_2, u_1$, and $u_2$ are the same as in the unicycle-type vehicle, $x_3 \in (-\pi, \pi]$ is the angle of the vehicle's heading measured from the east, $\phi \in (-\frac{\pi}{2}, \frac{\pi}{2})$ is the steering wheel angle relative to the vehicle body, and $l$ is the length between the center of the front wheels and the center of the back wheels. We set $l = 1$ for simplicity. As before, we set $u_1 = 1$ and let $u = u_2$. The above equations are rewritten as

$$\begin{aligned} \dot{x_1} &= \cos x_3 \\ \dot{x_2} &= \sin x_3 \\ \dot{x_3} &= \tan \phi \\ \dot{\phi} &= u. \end{aligned} \tag{8.6}$$

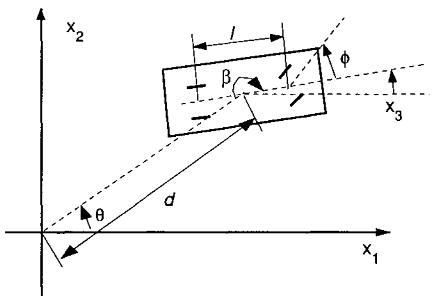


Figure 8.6 Kinematic model of a car-like vehicle.

### 8.2.2 Quantized Line Tracking

In this subsection, we suppose that system (8.6) is required to track the $x_1$ axis and point due east. Since there is no requirement on $x_1$, we can consider the following model:

$$\begin{aligned} \dot{x_2} &= \sin x_3 \\ \dot{x_3} &= \tan \phi \\ \dot{\phi} &= u. \end{aligned} \tag{8.7}$$

We need to stabilize $x_2, x_3$ and $\phi$. To find a CLF, we use back-stepping. For the dynamics of $x_2$ and $x_3$, let $V_1 = x_2^2 + x_3^2 + x_2 x_3$; then $\dot{V_1} = \sin x_3(2x_2 + x_3) + \tan \phi(2x_3 + x_2)$. $\dot{V_1}$ can be made negative by choosing $\tan \phi = -2 \sin x_3 - x_2 - 2x_3$.

Let $V = V_1 + (\phi + \arctan(2\sin x_3 + x_2 + 2x_3))^2$. Then $L_gV = 2(\phi + \arctan(2\sin x_3 + x_2 + 2x_3))$, and $L_fV = \dot{V_1} + \frac{L_gV(2\cos x_3 \tan\phi + \sin x_3 + \tan\phi)}{1+(2\sin x_3 + x_2 + 2x_3)^2}$. We can verify that $V$ is indeed a CLF, and moreover, an RCLF. Direct calculation shows that $\alpha_M^2 = 0.0116$. Thus we can design a quantizer based on Proposition A.1.

**Quantizer Design 8.3.** *System (8.7) is robustly stabilized by a $\rho$-based SLQ with* $\rho = \frac{1-\sqrt{1-\alpha^2/\alpha_M^2}}{1+\sqrt{1-\alpha^2/\alpha_M^2}}$, $p(x) = 2(\phi + \arctan(2\sin x_3 + x_2 + 2x_3))$, $u_0 = \frac{1}{2\alpha^2}(\sqrt{1-\alpha^2/\alpha_M^2} - 1)\gamma_0$, *and* $0 < \alpha^2 < \alpha_M^2 = 0.0116$.

In Figure 8.7, we choose $\alpha^2 = 0.01$, and the vehicle starts from $x_1 = 0$, $x_2 = 2$. The initial posture is given by $x_3 = 2.1$ (northwest), and $\phi = 1$, so the front wheels are pointing west. **A** shows the desired path (the dashed line) in the $x_1$-$x_2$ plane. **B** shows the control input, and **C** shows the values of $x_3$ and $\phi$. We see that the vehicle follows a natural trajectory to reach the $x_1$ axis (with some overshoot), and then goes along it. The interaction between the quantizer and the vehicle is coarsely distributed along the time axis.

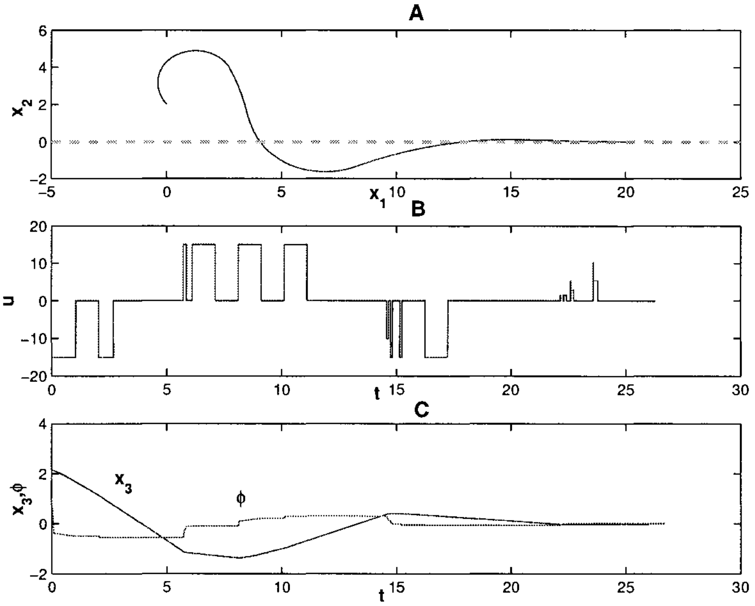


Figure 8.7 Sample trajectory for the motion of a car-like vehicle.

### 8.2.3 Quantized Position Control

In this subsection, we suppose that system (8.6) is required to reach the origin of the $x_1$-$x_2$ plane. We get

$$\begin{aligned} \dot{d} &= -\cos\beta \\ \dot{\beta} &= \tfrac{1}{d}\sin\beta - \tan\phi \\ \dot{\phi} &= u. \end{aligned} \tag{8.8}$$

by following the same procedure and coordinate transformation as in the last section, with the exception that the deviation angle $\beta$ is now given by $\beta = \theta - x_3 \pm \pi$.

The quantized position control of a car-like vehicle can be obtained by using the same approach as described before. We can use back-stepping to find a CLF, and $\beta$ and $\phi$ can be driven to zero. Let $V_1 = \beta^2$; $\dot{V}_1$ can be made negative by choosing $\tan\phi = \frac{2}{d}\sin\beta$. Then the smooth CLF for (8.8) is given by

$$V = V_1 + (\phi - \arctan(\frac{2}{d}\sin\beta))^2.$$

So we have

$$L_gV = 2(\phi - \arctan(\frac{2}{d}\sin\beta))$$

and

$$L_fV = (2\beta - L_gV\frac{2\cos\beta}{d(1+(\frac{2}{d}\sin\beta)^2)})(\sin\beta/d - \tan\phi) - L_gV\frac{\sin 2\beta}{d^2(1+(\frac{2}{d}\sin\beta)^2)}.$$

With this CLF, we can apply the results of Chapter 6 to design a quantizer. We skip the quantizer design procedure here since it resembles the quantizer obtained in Section 8.1.3. We present only the simulation results here.

Figure 8.8 shows sample trajectories for the motion of a car-like vehicle. The initial positions of trajectories $A, B$, and $C$ are $x_1 = 20\sqrt{2}$, $x_2 = 20\sqrt{2}$. The initial postures are given by $x_{3A} = 3\pi/4$, $\phi_A = -0.1$; $x_{3B} = \pi/4$, $\phi_B = -0.5$; $x_{3C} = \pi/5$, $\phi_C = 0.3$. We let $d^* = 0.2$, so the vehicle will reach a neighborhood of the destination with radius 0.2. As we see from the figure, the vehicle is steered to the required neighborhood.

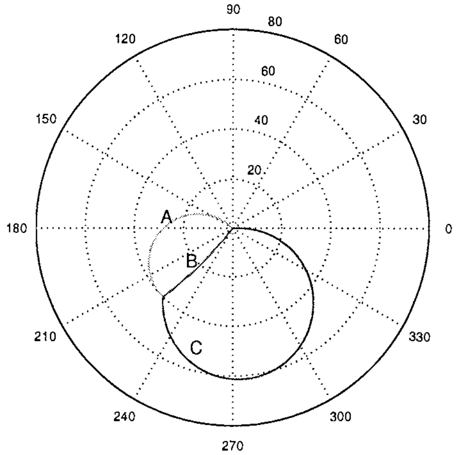


Figure 8.8 Sample trajectories for the motion of a car-like vehicle in polar coordinates.

## 8.3 Summary

In this chapter, we have presented several simulation results by applying the quantized control scheme established in preceding chapters. We have shown that for two types of simple vehicles, we can design a quantized control law so that the systems achieve the desired control objectives. The interaction between the quantizers and the vehicles is much reduced compared with traditional control schemes (control schemes that are not based on quantization).

# 9 CONCLUSIONS AND FUTURE RESEARCH

## 9.1 Conclusions

In the past few years, considerable efforts have been devoted to the study of quantized control systems, due to their theoretical and practical importance in the study of hybrid control systems, control under communication/computation constraints, the interaction between control and information, etc.

Recent papers [16, 13] establish quantized stabilization theory for single-input linear systems. This thesis is a direct extension of these results to single-input nonlinear affine systems.

First, we have shown that a single-input nonlinear affine system can be robustly stabilized by quantized feedback if it admits an RCLF. These robustly stabilizing quantizers have been explicitly constructed. Moreover, for a given RCLP, the coarsest robustly stabilizing quantizer has been shown to be essentially semi-logarithmic. Several important classes of control-affine systems, such as linear systems, feedback linearizable systems, and locally feedback linearizable systems have been shown to admit RCLF's, and their quantization has been examined.

Second, we have investigated the stabilizing quantizer for a single-input nonlinear affine system that admits a CLF. The designed quantizers in the closed-loop can be viewed as hierarchical hybrid automata. The quantized control strategy leads to a general control architecture for all single-input nonlinear affine systems with CLF's.

Furthermore, we have shown that the quantization-based control architecture is helpful in reducing the interaction between the controller and the system being controlled. We have also discussed how to eliminate chattering in a quantized control system. We have also derived preliminary results on quantization for partially feedback linearizable systems.

Finally, we have applied our theoretical results to several control system models. We have designed quantized controllers for two types of simple vehicles and achieved the desired control results.

To summarize, we have established results on quantized stabilization of nonlinear affine systems. Sufficient conditions that ensure the existence of quantized (robustly) stabilizing control have been found, the coarsest quantizers under certain conditions have been characterized, and a variety of quantizers have been constructed explicitly. Simulations demonstrated the advantages of the quantized control scheme.

## 9.2 Future Research

Research will continue on the following aspects:

1. An efficient way to solve the optimization problem of $\alpha_M^2$ in Lemma 3.3. We have seen that RCLF's are crucial in deriving "nice" quantizers. The problem of determining whether a CLF is an RCLF, and the problem of finding a suitable robustness level $\alpha$ depend heavily on this optimization problem. For linear systems, the problem is solved completely; however, for nonlinear systems, this is still an open problem.

2. Quantized stabilization of multi-input systems. Although the approach in this thesis extends to multi-input systems (which the reader can verify), the resulting quantizers are not in general the coarsest. In fact, multi-input systems involve much more complication than single-input systems, as is shown in [14] for two-input linear systems.

3. Conditions that guarantee the existence of stabilizing quantizers. We show in this thesis that, if a control-affine system admits a stabilizing regular feedback, then it admits a stabilizing quantized feedback. Some natural questions arise: Can

quantized feedback stabilize systems that regular feedback cannot stabilize? What are the conditions that guarantee the existence of quantized stabilizing feedback? Is "asymptotic controllability" equivalent to "quantized stabilizability"? In what sense should the solutions to quantized dynamics systems be interpreted?

4. Quantized performance. Quantization may lead to degeneration of performance; therefore, we need to investigate the balance between less control effort and the performance of the system, as is done in [7].

5. Control under communication/computation constraints. In this thesis, these constraints are not considered explicitly. It would be interesting to extend previous results (e.g. [52, 44]) on linear systems to nonlinear affine systems.

# APPENDIX A PROOFS OF RESULTS IN SECTION 5.1

First, we can easily show that if a robustly stabilizing quantizer is not symmetric in the $p(x) - u$ plane, then there exists a symmetric robustly stabilizing quantizer with the same density. (This can be done by symmetrically mapping one half of the graph into another half; we skip the details here.) Therefore, we can consider only symmetric quantizers for simplicity and without loss of generality.

Second, we can show that the choice of $\gamma_0$ is immaterial in considering robust stabilization problem (see Remark 3.1). Let $\Gamma = \{\gamma_0 \rho^i, i \in \mathbb{Z}\}$. The next lemma says that if for some SLQ $q \in \mathcal{Q}_\alpha(V)$ with $1 \in \Gamma$, then there exists some ESLQ $q_1 \in \mathcal{Q}_\alpha(V)$ with $\beta \in \Gamma$ for any $\beta > 0$, and $q_1$ has the same density as $q$.

**Lemma A.1.** *Suppose $(q, S, \Omega, \mathcal{U})$ with $1 \in \Gamma$ is a $\rho$-based SLQ in $\mathcal{Q}_\alpha(V)$. Then for any given $\beta > 0$, there exists a $\rho$-based ESLQ $q_1 \in \mathcal{Q}_\alpha(V)$ with $\beta \in \Gamma$, and $q_1$ has the same partition function as $q$.*

*Proof.* From Lemma 3.3, we know that for any given weight $0 < \alpha^2 < \alpha_M^2$, there must exist some $u$ for each $x \in S \setminus \{0\}$, such that (3.10) holds; i.e.,

$$\alpha^2 u^2 + L_g V(x) u + L_f V(x) < 0. \tag{A.1}$$

Let $U_\alpha$ be the set of pairs $(x, u)$ such that (A.1) holds; i.e., $U_\alpha = \{(x,u) | \alpha^2 u^2 + L_g V u + L_f V < 0, x \in S, x \neq 0\}$. Direct calculation shows that the boundaries of $U_\alpha$ are

$$u^{(1),(2)}(x) = \frac{1}{2\alpha^2}(-L_g V(x) \pm \sqrt{(L_g V(x))^2 - 4\alpha^2 L_f V(x)}\,). \tag{A.2}$$

Since every point between the two boundaries $u^{(1),(2)}(x)$ can be such that (3.10) holds, we know $U_\alpha$ can be characterized by its two boundaries. Therefore, $(x, u)$ is such that (3.10) holds if and only if $u$ is between the two boundaries $u^{(1),(2)}(x)$.

Suppose the partition function is $p(x)$. For any $u_i \in \mathcal{U}$ and any $x \in \Omega_i^+$, the pair $(x, u_i)$ is in $\mathrm{cl}(U_\alpha)$. Therefore, for all $x$ such $p(x) = \rho^{i+1}$ (i.e., $x$ on the boundary of $\Omega_i^+$ and $\Omega_{i+1}^+$), we have $(x, u_i) \in \mathrm{cl}(U_\alpha)$ and $(x, u_{i+1}) \in \mathrm{cl}(U_\alpha)$, which implies

$$\rho = \frac{u_{i+1}}{u_i} \geq \frac{u^{(1)}(x)}{u^{(2)}(x)}.$$

Direct calculation shows that the above equation is equivalent to

$$h(x) \geq \frac{\alpha^2}{1 - (\frac{1-\rho}{1+\rho})^2} \triangleq r$$

for all $x$ such that $p(x) = \rho^{i+1}$. Let $\underline{h}(y) = \inf_{p(x)=y} h(x)$. Then in any small neighborhood of 0 we have $\underline{h}(y) \geq r$ for some $y > 0$.

Note that to show the existence of the ESLQ $q_1$, we only need to show the existence of a $\rho$-based SLQ $q_2 \in \mathcal{Q}_\alpha(V)$ with $\beta \in \Gamma$ in a small neighborhood of the origin. Suppose such a $q_2$ does not exist; i.e., any $\rho$-based SLQ $q_2$ with $\beta \in \Gamma$ is not in $\mathcal{Q}_\alpha(V)$. Hence, in any small neighborhood of the origin, there exists some $x$ such that

$$\rho = \frac{u_{i+1}}{u_i} < \frac{u^{(1)}(x)}{u^{(2)}(x)},$$

which means in any small neighborhood of 0 we have $\underline{h}(y) < r$ for some $y > 0$.

However, in a small enough neighborhood $N$ of 0, function $\underline{h}(y)$ is monotonic or constant. In fact, on $N$ we know $h(\lambda x)$ is a monotonic or constant function of $\lambda$, where $\lambda > 0$. Thus, it is impossible that $\underline{h}(p(x))$ achieves its minimum (or maximum) at some $x$ in the interior of $N \backslash \{0\}$; otherwise the minimum (or maximum) could be made smaller (or larger) by scaling $x$. Thus, $\underline{h}(y)$ is monotonic or constant, which makes it impossible that in any small neighborhood of 0, we have $\underline{h}(y) < r$ for some $y > 0$ and $\underline{h}(y) \geq r$ for some other $y > 0$. So, there must exist a $\rho$-based SLQ $q_2 \in \mathcal{Q}_\alpha(V)$ with $\beta \in \Gamma$ in a small neighborhood of the origin. □

Now we prove a proposition which constructs the coarsest SLQ $q \in \mathcal{Q}_\alpha(V)$.

**Proposition A.1.** *Suppose $(V(x), \alpha)$ is an RCLP for system (3.1) on $S$, and $\alpha_S^2 = \alpha_M^2$. System (3.1) can be robustly stabilized to the origin by a $\rho_c$-based semi-logarithmic quantizer $q_c$ on $S$ with the partition function $p(x) = L_g V(x)$, $\rho_c = k_{1c}/k_{2c} < 1$, $k_{1c} = \frac{-1+\sqrt{1-\alpha^2/\alpha_M^2}}{2\alpha^2}$, $k_{2c} = \frac{-1-\sqrt{1-\alpha^2/\alpha_M^2}}{2\alpha^2}$, and $u_0 = k_{1c}\gamma_0$. $q_c$ is the coarsest SLQ in $\mathcal{Q}_\alpha(V)$.*

*Proof.* Let $U_\alpha = \{(x,u)|\alpha^2u^2 + L_gVu + L_fV < 0, x \in S, x \neq 0\}$. The boundaries of $U_\alpha$ are given by equation A.2. We know that any function that lies between the two boundaries is a candidate for robustly stabilizing feedback. We want to find an SLQ lying between the boundaries. Such an SLQ is characterized as follows.

**Claim:** Let

$$\hat{u}^{(1),(2)} = \frac{-1 \pm \sqrt{1-\alpha^2/\alpha_M^2}}{2\alpha^2} L_gV, \tag{A.3}$$

and $\hat{U}_\alpha = \{(x,u)|\hat{u}^{(1)}(x) > u > \hat{u}^{(2)}(x)\}$. Then $\hat{U}_\alpha \subseteq U_\alpha$, and $\hat{U}_\alpha$ totally contains any semi-logarithmic quantization in the set $U_\alpha$.

**Proof of Claim:** The first part of the proof of the claim is done by direct verification. For the second part, we only need to show that the coarsest SLQ in the set $U_\alpha$ is inside $\hat{U}_\alpha$. By Lemma A.1 and the assumption $\alpha_S^2 = \alpha_M^2$, we know that an SLQ is in $U_\alpha$ if and only if its triggering manifolds are in $U_\alpha$. Let us assume that the two triggering manifolds of the coarsest SLQ $q$ are $c_1p(x)$ and $c_2p(x)$, where $p(x)$ is some smooth function, and $c_1, c_2$ are constants. Since $|u^{(1)} - \frac{-L_gV}{2\alpha^2}| = |u^{(2)} - \frac{-L_gV}{2\alpha^2}|$, we have $|c_1p(x) - \frac{-L_gV}{2\alpha^2}| = |c_2p(x) - \frac{-L_gV}{2\alpha^2}|$; otherwise the two triggering manifolds could be made closer to $u^{(1),(2)}$, which contradicts the fact that $q$ is the coarsest. So

$$p(x) = \frac{-L_gV}{\alpha^2(c_1+c_2)}. \tag{A.4}$$

Then direct calculation shows that $c_1p(x) = k_{1c}L_gV$ and $c_2p(x) = k_{2c}L_gV$, where $k_{1c}$ and $k_{1c}$ are defined as in the hypothesis of Proposition A.1. This proves the claim. $\nabla$

By virtue of the claim, we can do quantization for system (3.1). For the compact set $S \subseteq X$ containing the origin, we know $x \neq 0$ implies $\alpha^2q_c^2 + L_gVq_c + L_fV < 0$. Then according to Lemma 3.5, $q_c$ is a robustly stabilizing quantizer on $S$, and it is the coarsest semi-logarithmic one in $\mathcal{Q}_\alpha(V)$. $\square$

The above proposition gives us the coarsest SLQ for a given RCLP. The next lemma helps us to characterize the coarsest quantizer for a given RCLP.

**Lemma A.2.** *Let the RCLP $(V(x), \alpha)$ and the SLQ $q_c$ be given as in Proposition A.1, and $\alpha_M^2 = \alpha_S^2$. Then $q_c \in \arg\inf_{g\in\mathcal{Q}_\alpha(V)}\eta_g$.*

*Proof.* We can first show that, for any quantizer $q \in \mathcal{Q}_\alpha(V)$, it holds that for each $i \in \mathbb{Z}$, there exists some $x$ such that

$$\frac{u_{i+1}}{u_i} \geq \frac{u^{(1)}(x)}{u^{(2)}(x)}.$$

Otherwise, suppose for some $i$ it holds that

$$\frac{u_{i+1}}{u_i} < \frac{u^{(1)}(x)}{u^{(2)}(x)}$$

for all $x$, then it implies that there must exist some $x$ such that for all $u \geq u_i$, we have

$$\alpha^2 u^2 + L_g V(x) u + L_f V(x) > 0,$$

and for all $u \leq u_{i+1}$, we have

$$\alpha^2 u^2 + L_g V(x) u + L_f V(x) > 0.$$

This leads to contradiction.

Then we can show that

$$\frac{u_{i+1}}{u_i} \geq \rho_c \triangleq e^{-\frac{1}{\eta_{qc}}} = \frac{k_{1c}}{k_{2c}}$$

for all $i$. This is easily shown by noticing

$$\sup_x \frac{u^{(1)}(x)}{u^{(2)}(x)} = \frac{k_{1c}}{k_{2c}}.$$

Therefore, we know that all density smaller $\eta_c$ is not achievable. □

Now we prove Proposition 5.1 using the results we just showed.

**Proof of Proposition 5.1**

*Proof.* Suppose there is some $a > 0$ such that $\alpha_S^2 \leq h(x)$ for all $x$ in $S_{f_a} = \{x \in S_f | V(x) \leq a\}$. Let $S_a = \{x \in S | V(x) \leq a\}$. Define $\alpha_{Ma}^2 = \inf_{x \in S_{f_a}} h(x)$. It is easy to see that $\alpha_{Ma}^2 = \alpha_S^2$.

By Proposition A.1, we know that on the compact set $S_a$, the coarsest SLQ $q_c \in \mathcal{Q}_\alpha(V)$ is given as $p(x) = L_g V(x)$, $\rho_c = k_{1c}/k_{2c} < 1$, $k_{1c} = \frac{-1+\sqrt{1-\alpha^2/\alpha_{Ma}^2}}{2\alpha^2}$, $k_{2c} = \frac{-1-\sqrt{1-\alpha^2/\alpha_{Ma}^2}}{2\alpha^2}$, and $u_0 = k_{1c}\gamma_0$, which is in fact the semi-logarithmic part of the ESLQ defined in Proposition 5.1. The finite number of cells on $S \setminus S_a$ drive the state from $S \setminus S_a$ into $S_a$, as ensured by Proposition 5.3. Because $q_c$ and $q^*$ differ only on a finite number of cells, we know they have the same density. □

### Proof of Proposition 5.2

*Proof.* For the given $\epsilon > 0$, let $\eta_\epsilon = \eta^* + \epsilon$, $\rho_\epsilon = e^{-\frac{1}{\eta_\epsilon}}$, $\alpha_\epsilon^2 = \frac{\alpha^2}{1-(\frac{1-\rho_\epsilon}{1+\rho_\epsilon})^2}$, $S_{f\epsilon} = \{x \in S_f | h(x) > \alpha_\epsilon^2\}$, and $S_\epsilon$ be the smallest closed invariant set of $V(x)$ containing $S_{f\epsilon}$. Define $\alpha_{M\epsilon}^2 = \inf_{x \in S_{f\epsilon}} h(x)$. It is easy to see that $\alpha_{M\epsilon}^2 = \alpha_\epsilon^2$.

By Proposition A.1, we know that on the compact set $S_\epsilon$, the coarsest SLQ $q_c \in \mathcal{Q}_\alpha(V)$ is given as $p(x) = L_g V(x)$, $\rho_c = k_{1c}/k_{2c} < 1$, $k_{1c} = \frac{-1+\sqrt{1-\alpha^2/\alpha_{M\epsilon}^2}}{2\alpha^2}$, $k_{2c} = \frac{-1-\sqrt{1-\alpha^2/\alpha_{M\epsilon}^2}}{2\alpha^2}$, and $u_0 = k_{1c}\gamma_0$, which is in fact the semi-logarithmic part of the ESLQ defined in the hypothesis of Proposition 5.2. Because $q_c$ and $q_\epsilon$ differ only on a finite number of cells, we know they have the same density. □

### Proof of Lemma 5.1

*Proof.* First we show that the condition is sufficient. Suppose there is some $a > 0$ such that $\alpha_S^2 \le h(x)$ for all $x$ in $S_{f_a} = \{x \in S_f | V(x) \le a\}$. By Proposition 5.1, we know $\eta^*$ is an achieved density of robustly stabilizing quantizers. So $q^* \in \mathcal{Q}_\alpha(V)$.

Next we show that the condition is also necessary. We prove by contradiction. Suppose for any $a > 0$, there exists some $x$ in $S_{f_a} = \{x \in S_f | V(x) \le a\}$ such that $\alpha_S^2 > h(x)$, but $q^* \in \mathcal{Q}_\alpha(V)$. By a reasoning similar to that in the proof of Lemma A.1, we know $q^* \in \mathcal{Q}_\alpha(V)$ implies that

$$h(x) \ge \frac{\alpha^2}{1-(\frac{1-\rho}{1+\rho})^2} = \alpha_S^2$$

for all $x \in S_f$ such that $L_g V(x) = \rho^i$ for all $i$. Let $\underline{h}(y) = \inf_{L_g V(x)=y} h(x)$. Then in any small neighborhood of 0 it holds $\underline{h}(y) \ge \alpha_S^2$ for some $y > 0$. However, in any small neighborhood of 0 it holds also that $\alpha_S^2 > h(x)$ for some $x$, and hence $\underline{h}(y) < \alpha_S^2$ for some $y > 0$, which is a contradiction. □

### Proof of Theorem 5.1

*Proof.* By Proposition 5.2 and Lemma A.2, we know $\eta^*$ is the infimum of the density of robustly stabilizing quantizers. The case of $\alpha_S^2 = +\infty$ is actually the limiting case of $\alpha_S^2 < +\infty$. □

# APPENDIX B PROOFS OF RESULTS IN SECTION 5.3

## Proof of Lemma 5.2

*Proof.* First we show $\sigma^2 = B'PQ^{-1}PB > 1$. Notice $\sigma$ is the singular value of $B'PQ^{-\frac{1}{2}}$, so the SVD (Singular Value Decomposition) for $B'PQ^{-\frac{1}{2}}$ has the following form:

$$B'PQ^{-\frac{1}{2}} = S_1 \Sigma S_2$$

where $S_1 = 1 \in \mathbb{R}$, $\Sigma = [\sigma, 0, \ldots, 0] \in \mathbb{R}^{1\times n}$, $S_2 \in \mathbb{R}^{n\times n}$ and $S_2' S_2 = I$.

So

$$\begin{aligned} & S_2 \Sigma' \Sigma S_2' \\ &= Q^{-\frac{1}{2}} PBB'PQ^{-\frac{1}{2}} \\ &= Q^{-\frac{1}{2}} (PA + A'P + Q) Q^{-\frac{1}{2}}, \end{aligned} \tag{B.1}$$

which yields

$$\begin{aligned} & PA + A'P \\ &= Q^{\frac{1}{2}} S_2 \Sigma' \Sigma S_2' Q^{\frac{1}{2}} - Q \\ &= Q^{\frac{1}{2}} S_2 (\Sigma'\Sigma - I) S_2' Q^{\frac{1}{2}} \\ &= Q^{\frac{1}{2}} S_2 \mathrm{diag}(\sigma^2 - 1, -1, \ldots, -1) S_2' Q^{\frac{1}{2}}. \end{aligned} \tag{B.2}$$

By the result of classification of CLF's (refer to [15]), we know that if $A$ is not Hurwitz, $A'P + PA$ has one and only one positive eigenvalue, which implies that diagonal matrix $\mathrm{diag}(\sigma^2 - 1, -1, \ldots, -1)$ has one positive eigenvalue. Therefore, $\sigma^2 > 1$.

On the set $\{x | L_f V(x) > 0, x \in \mathbb{R}^n\}$, we have

$$\begin{aligned} & \tfrac{(L_g V)^2}{4 L_f V} \\ &= \tfrac{x'PBB'Px}{x'(A'P+PA)x} \\ &= \tfrac{1}{\frac{x'(A'P+PA)x}{x'PBB'Px}}. \end{aligned} \tag{B.3}$$

Notice that $A'P + PA - PBB'P = -Q$, so we know the above equals

$$\begin{aligned} &\frac{1}{\frac{x'PBB'Px - x'Qx}{x'PBB'Px}} \\ =&\frac{1}{1-\frac{x'Qx}{x'PBB'Px}}. \end{aligned} \tag{B.4}$$

So we need to calculate $\inf_x \dfrac{x'Qx}{x'PBB'Px}$. This is equivalent to solving

$$\max_{\frac{x'Qx}{x'PBB'Px} \geq t, \forall x} \quad t. \tag{B.5}$$

Note that

$$\begin{aligned} &\frac{x'Qx}{x'PBB'Px} \geq t \\ &\Leftrightarrow Q \geq tPBB'P \\ &\Leftrightarrow \frac{1}{t} I \geq Q^{-\frac{1}{2}} PBB'PQ^{-\frac{1}{2}} \\ &\Leftrightarrow \frac{1}{t} \geq B'PQ^{-\frac{1}{2}} Q^{-\frac{1}{2}} PB. \end{aligned} \tag{B.6}$$

So the optimal solution of (B.5) is $\frac{1}{\sigma^2}$. Note that $\alpha_S^2 = \alpha_M^2$ since $h(x)$ is invariant w.r.t. scaling. Then the result follows. □

**Proof of Proposition 5.4**

*Proof.* The proof is straightforward given Lemma 5.2. □

**Proof of Proposition 5.5**

*Proof.* Since $V_M = 1$ and $V = x'Px$, we have $DV = Px$ and hence

$$x'P^2x \leq 1$$

for any $x$ in $S$; i.e.,

$$x'P^2x \leq 1 \text{ for any } x \quad s.t. \quad x'x \leq 1. \tag{B.7}$$

Using the $\mathcal{S}$-procedure (see [3]), (B.7) is equivalent to $P \leq I$.

Therefore, for the given robustness level $\alpha$, the searching of the coarsest quantizer is equivalent to the following optimization problem:

$$\begin{aligned} \inf \quad & \sigma^2 \\ s.t.\ & \sigma^2 = B'PQ^{-1}PB \\ & 0 < P \leq I \\ = \inf \quad & \sigma^2 \\ s.t.\ & \sigma^2 \geq B'PQ^{-1}PB \\ & 0 < P \leq I \end{aligned} \tag{B.8}$$

Noticing $\sigma^2 \geq B'PQ^{-1}PB$ if and only if $\sigma^2 Q \geq PBB'P$ (similar to the reasoning in equation (B.6)), we know the above optimization problem is equivalent to

$$\begin{aligned} \inf \quad & \sigma^2 \\ s.t.\ & \sigma^2 Q \geq PBB'P \\ & 0 < P \leq I \\ = \inf \quad & \sigma^2 \\ s.t.\ & A'P + PA - (1 - \tfrac{1}{\sigma^2})PBB'P < 0 \\ & 0 < P \leq I \\ = \inf \quad & \gamma \\ s.t.\ & RA' + AR - \gamma BB' < 0 \\ & R \geq I \\ & \gamma < 1 \end{aligned} \tag{B.9}$$

where $\gamma = (1 - \frac{1}{\sigma^2})$, and $RP = I$. □

# APPENDIX C PROOFS OF RESULTS IN SECTION 5.4

Firstly, we present two ways to prove Lemma 5.3. The first one, although lengthy, is helpful to address partially feedback linearizable systems; the second one is concise and relies on the results in [26]. Then we show Proposition 5.6 is a direct result of the lemma.

**Proof 1 of Lemma 5.3**

*Proof.* (a) First we can show $V$ is a CLF on $S$ for system (5.6). For affine system $\dot{x} = f(x) + g(x)u$ we only need to show that, if $L_gV = 0$ and $x \neq 0$, then $L_fV < 0$. Here we have $L_gV(x) = 2\beta^{-1}(x)B'Px$, and $L_fV(x) = x'(A'P + PA)x - \gamma(x)L_gV(x)$. So $L_gV(x) = 0$ if and only if $B'Px = 0$. $V$ is a CLF for the linear system $\dot{x} = Ax + Bu$, so $B'Px = 0$ implies $x'(A'P + PA)x < 0$. Therefore $L_gV(x) = 0$ and $x \neq 0$ implies $L_fV(x) < 0$.

Now we show that $V$ is an RCLF on $S$; that is, for some $\alpha^2 > 0$, there is some $u_x$ such that for each $0 \neq x \in S$, we have

$$\alpha^2 u_x^2 + 2\beta^{-1}(x)B'Pxu_x + x'(A'P + PA)x - 2\gamma(x)\beta^{-1}(x)B'Px < 0. \tag{C.1}$$

By the previous result, we know $V$ is an RCLF if and only if

$$\alpha_M^2 = \inf_{\substack{x \in S \\ L_fV > 0}} \frac{(L_gV)^2}{4L_fV} > 0. \tag{C.2}$$

If $L_fV > 0$, then

$$\begin{aligned} & x'(A'P + PA)x - \gamma(x)L_gV(x) \\ & = x'(PBB'P - Q)x - \gamma(x)L_gV(x) \\ & > 0. \end{aligned} \tag{C.3}$$

So $x'PBB'Px > x'Qx + \gamma(x)L_gV(x)$, and it yields that

$$\frac{x'Qx + \gamma(x)L_gV(x)}{x'PBB'Px} < 1.$$

We can furthermore show that there exists an $M > 0$ such that

$$\frac{x'Qx + \gamma(x)L_gV(x)}{x'PBB'Px} \geq -M; \tag{C.4}$$

that is, the fraction is bounded from below. Notice $x'PBB'Px \neq 0$ if $L_fV > 0$, so we only need to show there exist an $M > 0$ such that

$$x'Qx + 2\gamma(x)\beta^{-1}(x)B'Px + Mx'PBB'Px \geq 0. \tag{C.5}$$

By smoothness of $\gamma(x)\beta^{-1}(x)$, and because $\gamma(0)\beta^{-1}(0) = 0$, the Mean Value Theorem implies that there is some $x_0 \in S$ such that

$$2\gamma(x)\beta^{-1}(x) = H'(x_0)x, \tag{C.6}$$

where $H(x) : S \rightarrow \in \mathbb{R}^n$ is the gradient of the LHS function. So (C.5) is equivalent to

$$x'Qx + H'(x_0)xB'Px + Mx'PBB'Px \geq 0, \tag{C.7}$$

which is

$$x'(Q + (H(x_0) + MPB)B'P)x \geq 0. \tag{C.8}$$

Suppose $H(x_0) = (h_1(x_0), h_2(x_0), \ldots, h_n(x_0))'$. Each component in $H(x_0)$ takes a finite value over the compact set $S$, so for each scalar function $|h_i(x_0)|$ we can find its supremum over $S$, say, $h_i$. Assume without loss of generality that $(A, B)$ is in controllable canonical form, so $B = (0, 0, \ldots, 0, 1)'$. Assume $P = \{p_{ij}\}$. Then $B'P = (p_{1n}, p_{2n}, \ldots, p_{nn})$.

If $p_{in} = 0$ for some $i$, then $Mp_{in}^2 + p_{in}h_i(x) = 0$ for any $M$. Next, choose $M$ such that $M > \frac{h_i}{|p_{in}|}$ for all $i$ such that $p_{in} \neq 0$, so we have

$$\begin{aligned} &Mp_{in}^2 + p_{in}h_i(x_0) \\ &\geq Mp_{in}^2 - |p_{in}|h_i \\ &\geq (M|p_{in}| - h_i)|p_{in}| \\ &> 0. \end{aligned} \tag{C.9}$$

In both cases we conclude that there is an $M > 0$ such that $Mp_{in}^2 + p_{in}h_i(x) \geq 0$ for all $i$.

Therefore we have

$$\begin{aligned}
&B'P(H(x_0) + MPB) \\
&= (p_{1n}, p_{2n}, \ldots, p_{nn})(h_1(x_0) + Mp_{1n}, h_2(x_0) + Mp_{2n}, \ldots, h_n(x_0) + Mp_{nn})' \\
&= (Mp_{1n}^2 + p_{1n}h_1(x_0)) + (Mp_{2n}^2 + p_{2n}h_n(x_0)) + \ldots + (Mp_{nn}^2 + p_{nn}h_n(x_0)) \\
&> 0.
\end{aligned} \tag{C.10}$$

Note that ">" holds here because $P \succ 0$ ($P$ being positive definite) and hence $p_{nn} > 0$.

The only possible nonzero eigenvalue of matrix $\check{H} = (H(x_0) + MPB)B'P$ is $B'P(H(x_0) + MPB)$, so $\check{H} \succeq 0$ ($\check{H}$ being positive semi-definite). Thus (C.8) holds and consequently (C.4) holds.

Since

$$\frac{(L_gV)^2}{4L_fV} = \beta^{-2}(x)\frac{1}{1 - \frac{x'Qx + \gamma(x)L_gV(x)}{x'PBB'Px}},$$

we have

$$\frac{(L_gV)^2}{4L_fV} > \beta^{-2}(x) \times \frac{1}{1+M}$$

and thus $\alpha_M^2 > 0$.

(b) By (a), we know the $\alpha_S^2$ for the QCLF $V$ is positive. It is easily verified that adding the higher order term $V_h$ to $V$ does not change $\alpha_S^2$, and hence $V_s$ is an RCLF. □

### Proof 2 of Lemma 5.3

*Proof.* Let $J_\gamma$ be the Jacobian matrix of $\gamma(x)$ evaluated at $x = 0$; i.e., $J_\gamma = \frac{\partial \gamma}{\partial x}|_{x=0}$. Let $J$ be the Jacobian matrix of $\beta^{-1}(x)\gamma(x)$ evaluated at $x = 0$; i.e.,

$$\begin{aligned}
J &= \tfrac{\partial(\beta^{-1}(x)\gamma(x))}{\partial x}|_{x=0} \\
&= \tfrac{\partial \beta^{-1}(x)}{\partial x}\gamma(x)|_{x=0} + \beta^{-1}(x)\tfrac{\partial \gamma(x)}{\partial x}|_{x=0} \\
&= \beta^{-1}(0)J_\gamma.
\end{aligned} \tag{C.11}$$

Then we rewrite system (5.6) as follows:

$$\begin{aligned}\dot{x} &= Ax + B\beta^{-1}(x)(u - \gamma(x)) \\ &= Ax - B\beta^{-1}(x)\gamma(x) + B\beta^{-1}(x)u \\ &= Ax - BJx + B\beta^{-1}(0)u + (BJx - B\beta^{-1}(x)\gamma(x)) + (B\beta^{-1}(x) - B\beta^{-1}(0))u\end{aligned} \tag{C.12}$$

where the last two terms of the last equality are higher order terms, and $(A-BJ, B\beta^{-1}(0))$ is the Jacobian linearization of system (5.6).

Let $\bar{B} = \beta^{-1}(0)B$, then

$$\begin{aligned}&(A - BJ, B\beta^{-1}(0)) \\ &= (A - B\beta^{-1}(0)J_\gamma,\ B\beta^{-1}(0)) \\ &= (A - \bar{B}J_\gamma,\ \bar{B}).\end{aligned} \tag{C.13}$$

By assumption we know $V = x'Px$ is a CLF for $(A, B)$, therefore $V$ is a CLF for $(A, \bar{B})$ (since $\beta^{-1}(0)$ is a scalar), and hence $V$ is a CLF for $(A - \bar{B}J_\gamma,\ \bar{B})$. Because $V$ is a QCLF for system (5.6) and its Jacobian linearization, by Corollary 1 in [26], we know $V$ is an RCLF for system (5.6). □

### Proof of Proposition 5.6

*Proof.* The proof is straightforward by using Lemma 5.3. □

# APPENDIX D PROOFS OF RESULTS IN SECTION 7.3

## Proof of Lemma 7.2

*Proof.* Suppose $V(x) = x'Px$ is a quadratic RCLF for system (7.4); that is, there exists $\alpha_1^2 > 0$ such that

$$\inf_{\substack{x \in S \\ L_{f1}V > 0}} \frac{(L_{g1}V)^2}{4L_{f1}V} > \alpha_1^2, \tag{D.1}$$

where $L_{f1} = \frac{\partial V}{\partial \xi}A\xi + \frac{\partial V}{\partial \eta}q(\xi,\eta)$, and $L_{g1}V = \frac{\partial V}{\partial \xi}B$. The above equation is equivalent to

$$(\frac{\partial V}{\partial \xi}B)^2 > 4\alpha_1^2(\frac{\partial V}{\partial \xi}A\xi + \frac{\partial V}{\partial \eta}q(\xi,\eta)) \tag{D.2}$$

for all $0 \neq (\xi,\eta) \in S$.

As a consequence, there exists a positive definite function $Q(x) : S \to \mathbb{R}$ such that

$$(\frac{\partial V}{\partial \xi}B)^2 = 4\alpha_1^2(\frac{\partial V}{\partial \xi}A\xi + \frac{\partial V}{\partial \eta}q(\xi,\eta)) + Q(x) \tag{D.3}$$

for all $(\xi,\eta) \in S$.

For system (7.2), we have

$$L_{f2}V = \frac{\partial V}{\partial \xi}A\xi - \frac{\partial V}{\partial \xi}B\beta^{-1}(\xi,\eta)\gamma(\xi,\eta) + \frac{\partial V}{\partial \eta}q(\xi,\eta)$$

and

$$L_{g2}V = \frac{\partial V}{\partial \xi}B\beta^{-1}(\xi,\eta).$$

To show $V(x)$ is an RCLF for (7.2), we need to show

$$\alpha_2^2 = \inf_{\substack{x \in S \\ L_{f2}V > 0}} \frac{(L_{g2}V)^2}{4L_{f2}V} > 0. \tag{D.4}$$

Because $\beta(x)$ is nonsingular for all $x$ in the compact set $S$, we need only to show there exists $\alpha_3 > 0$ such that

$$\frac{(\frac{\partial V}{\partial \xi}B)^2}{\frac{\partial V}{\partial \xi}A\xi - \frac{\partial V}{\partial \xi}B\beta^{-1}(\xi,\eta)\gamma(\xi,\eta) + \frac{\partial V}{\partial \eta}q(\xi,\eta)} > \alpha_3^2 \tag{D.5}$$

for all $(\xi, \eta)$ so that $L_{f2}V > 0$.

Note that

$$\begin{aligned} & \frac{(\frac{\partial V}{\partial \xi}B)^2}{\frac{\partial V}{\partial \xi}A\xi - \frac{\partial V}{\partial \xi}B\beta^{-1}(x)\gamma(x) + \frac{\partial V}{\partial \eta}q(x)} \\ = & \frac{(\frac{\partial V}{\partial \xi}B)^2}{\frac{(\frac{\partial V}{\partial \xi}B)^2 - Q(x)}{4\alpha_1^2} - \frac{\partial V}{\partial \xi}B\beta^{-1}(x)\gamma(x)} \\ = & \frac{1}{\frac{1}{4\alpha_1^2} - \frac{\frac{\partial V}{\partial \xi}B\beta^{-1}(x)\gamma(x) + \frac{Q(x)}{4\alpha_1^2}}{(\frac{\partial V}{\partial \xi}B)^2}}. \end{aligned} \tag{D.6}$$

Therefore, to show equation (D.5), it is sufficient to show

$$\frac{\frac{\partial V}{\partial \xi}B\beta^{-1}(x)\gamma(x) + \frac{Q(x)}{4\alpha_1^2}}{(\frac{\partial V}{\partial \xi}B)^2} \tag{D.7}$$

is bounded from below, which is true if there is some $M > 0$ such that for all $x \in S$,

$$\frac{Q(x)}{4\alpha_1^2} + \frac{\partial V}{\partial \xi}B\beta^{-1}(x)\gamma(x) + M(\frac{\partial V}{\partial \xi}B)^2 \geq 0. \tag{D.8}$$

Let $\bar{B} = (B', 0, \cdots, 0)' \in \mathbb{R}^n$. Then it is easy to check that $\frac{\partial V}{\partial \xi}B = 2\bar{B}'Px$. So equation (D.8) is equivalent to

$$\frac{Q(x)}{4\alpha_1^2} + 2\gamma(x)\beta^{-1}(x)\bar{B}'Px + 4Mx'P\bar{B}\bar{B}'Px \geq 0. \tag{D.9}$$

However, (D.9) can be shown using the same techniques we used to prove equation (C.5) with minor modification. Therefore, the result follows. □

## Proof of Proposition 7.1

*Proof.* The proposition follows directly from the above lemma. □

## Proof of Lemma 7.3

*Proof.* Suppose $V(x)$ is a CLF for system (7.2). Then $\dot{V} = L_f V(x) + L_g V(x) u$, where

$$L_f V(x) = \frac{\partial V}{\partial \xi} A\xi - \frac{\partial V}{\partial \xi} B\beta^{-1}(\xi,\eta)\gamma(\xi,\eta) + \frac{\partial V}{\partial \eta} q(\xi,\eta)$$

and

$$L_g V(x) = \frac{\partial V}{\partial \xi} B\beta^{-1}(\xi,\eta).$$

Since $V$ is a CLF, we know that $L_g V(x) = 0$, $x \neq 0$ implies that $L_f V(x) < 0$. Notice $\beta^{-1}(x) \neq 0$, so we have $\frac{\partial V}{\partial \xi} B = 0$, $x \neq 0$ implies $\frac{\partial V}{\partial \xi} A\xi + \frac{\partial V}{\partial \eta} q(\xi,\eta) < 0$. This further implies that $V(x)$ is a CLF for system (7.4). The proof for the other direction is similar. □

## Proof of Lemma 7.4

*Proof.* Since $q(\xi,\eta)$ is a smooth function, there is an $(n-r) \times r$ smooth matrix $G(\xi,\eta)$ such that

$$q(\xi,\eta) = q(0,\eta) + G(\xi,\eta)\xi \tag{D.10}$$

for all $(\xi,\eta)$. For instance, if we write $r(\lambda) \triangleq q(\lambda\xi,\eta)$, then from $r(1) = r(0) + \int_0^1 r'(\lambda)d\lambda$ we conclude that one choice is

$$G(\xi,\eta) \triangleq \int_0^1 \left.\frac{\partial q(\xi,\eta)}{\partial \xi}\right|_{(\lambda\xi,\eta)} d\lambda.$$

(See [48] p244.)

Since $\frac{dW}{d\eta}$ is smooth and $\frac{dW}{d\eta}|_0 = 0$, the Mean Value Theorem implies that there exists some point $\eta_0$ such that

$$\frac{dW}{d\eta} = H'(\eta_0)\eta, \tag{D.11}$$

where $H(\eta) \in \mathbb{R}^{n-r}$ is the gradient of the LHS function.

Consider $V = \xi' P \xi + c_2 W(\eta)$, $c_2 > 0$. Using $v = -\frac{1}{2} B' P \xi$, we have

$$\begin{aligned}
\dot{V} &= \xi'(A'P + PA)\xi + 2B'P\xi v + c_2 \tfrac{dW}{d\eta} q(\xi, \eta) \\
&= -\xi' Q \xi + c_2 \tfrac{dW}{d\eta} q(0, \eta) + c_2 \tfrac{dW}{d\eta} G(\xi, \eta)\xi \\
&< -\xi' Q \xi - c_2 c_1 \|\eta\|^2 + c_2 \eta' H(\eta_0) G(\xi, \eta)\xi \\
&= -(Q^{\frac{1}{2}}\xi - \tfrac{c_2}{2} Q^{-\frac{1}{2}} G'(\xi, \eta) H'(\eta_0)\eta)^2 + \tfrac{c_2^2}{4} \eta' H(\eta_0) G(\xi, \eta) G'(\xi, \eta) H'(\eta_0)\eta - c_2 c_1 \|\eta\|^2 \\
&\leq -(Q^{\frac{1}{2}}\xi - \tfrac{c_2}{2} Q^{-\frac{1}{2}} G'(\xi, \eta) H'(\eta_0)\eta)^2 + \tfrac{c_2^2}{4} c_3 \|\eta\|^2 - c_2 c_1 \|\eta\|^2 \\
&\leq -(Q^{\frac{1}{2}}\xi - \tfrac{c_2}{2} Q^{-\frac{1}{2}} G'(\xi, \eta) H'(\eta_0)\eta)^2.
\end{aligned} \tag{D.12}$$

Note $c_3 > 0$ exists since $H(\eta_0)G(\xi, \eta)G'(\xi, \eta)H'(\eta_0)$ can only take finite values, and the last inequality holds for sufficiently small $c_2$. Therefore, we know that if we choose a small enough $c_2$, $\dot{V} < 0$ for $(\xi, \eta) \neq 0$. □

**Proof of Lemma 7.5**

*Proof.* Since $\xi' P \xi$ is a quadratic CLF for the linear system $\dot{\xi} = A\xi + Bv$, by Lemma 5.2 it is also an RCLF; therefore, for some $\alpha > 0$, there exists some control law $v = v(\xi)$ such that

$$\alpha^2 v^2 + \xi'(A'P + PA)\xi + 2B'P\xi v < 0 \tag{D.13}$$

if $\xi \neq 0$.

Since $V_1$ is a CLF for system (7.4), and the same control law $v = v(\xi)$ is stabilizing, we have

$$\xi'(A'P + PA)\xi + 2B'P\xi v + c\frac{dW}{d\eta} q(\xi, \eta) < 0 \tag{D.14}$$

if $(\xi, \eta) \neq 0$.

Adding (D.13) and (D.14), we have

$$\alpha^2 v^2 + 2\xi'(A'P + PA)\xi + 4B'P\xi v + c\frac{dW}{d\eta} q(\xi, \eta) < 0 \tag{D.15}$$

if $(\xi, \eta) \neq 0$. So $V$ is an RCLF for system (7.4). □

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